

## REVIEW - Vectors

By using vectors and defining appropriate operations between them, physical laws can often be written in a simple form. Since we will making extensive use of vectors in Dynamics, we will summarize in this handout some of their important properties.

### Vectors

For our purposes we will think of a *vector* as a mathematical representation of a physical entity which has both *magnitude* and *direction* in a 3D space. Examples of physical vectors are forces, moments, and velocities. Geometrically, a vector can be represented as arrows. The length of the arrow represents its magnitude. Unless indicated otherwise, we shall assume that parallel translation does not change a vector, and we shall call the vectors satisfying this property, *free vectors*. Thus, two vectors are equal if and only if they are parallel, point in the same direction, and have equal length.

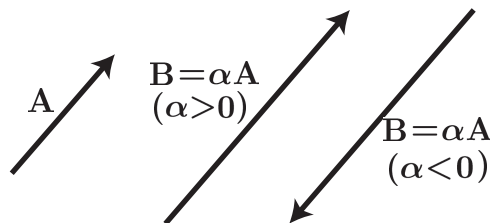
Vectors are usually typed in boldface and scalar quantities appear in lightface italic type, e.g. the vector quantity  $\mathbf{A}$  has magnitude, or modulus,  $A = |\mathbf{A}|$ . In handwritten text, vectors are often expressed using the arrow, or underbar notation, e.g.  $\vec{A}$ ,  $\underline{A}$ .

### Vector Algebra

Here, we introduce a few useful operations which are defined for free vectors.

#### Multiplication by a scalar

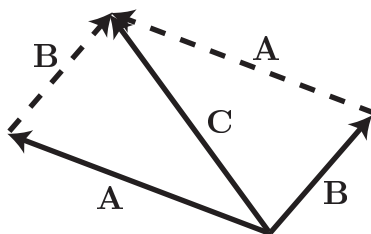
If we multiply a vector  $\mathbf{A}$  by a scalar  $\alpha$ , the result is a vector  $\mathbf{B} = \alpha\mathbf{A}$ , which has magnitude  $B = |\alpha|A$ . The vector  $\mathbf{B}$ , is parallel to  $\mathbf{A}$  and points in the same direction if  $\alpha > 0$ . For  $\alpha < 0$ , the vector  $\mathbf{B}$  is parallel to  $\mathbf{A}$  but points in the opposite direction (antiparallel).



If we multiply an arbitrary vector,  $\mathbf{A}$ , by the inverse of its magnitude,  $(1/A)$ , we obtain a *unit vector* which is parallel to  $\mathbf{A}$ . There exist several common notations to denote a unit vector, e.g.  $\hat{\mathbf{A}}$ ,  $\mathbf{e}_A$ , etc. Thus, we have that  $\hat{\mathbf{A}} = \mathbf{A}/A = \mathbf{A}/|\mathbf{A}|$ , and  $\mathbf{A} = A \hat{\mathbf{A}}$ ,  $|\hat{\mathbf{A}}| = 1$ .

## Vector addition

Vector addition has a very simple geometrical interpretation. To add vector  $\mathbf{B}$  to vector  $\mathbf{A}$ , we simply place the tail of  $\mathbf{B}$  at the head of  $\mathbf{A}$ . The sum is a vector  $\mathbf{C}$  from the tail of  $\mathbf{A}$  to the head of  $\mathbf{B}$ . Thus, we write  $\mathbf{C} = \mathbf{A} + \mathbf{B}$ . The same result is obtained if the roles of  $\mathbf{A}$  and  $\mathbf{B}$  are reversed. That is,  $\mathbf{C} = \mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$ . This commutative property is illustrated below with the parallelogram construction.

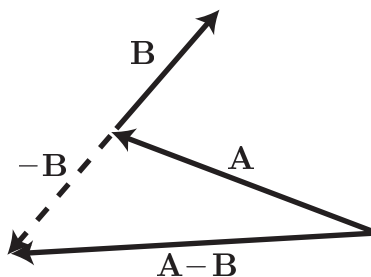


Since the result of adding two vectors is also a vector, we can consider the sum of multiple vectors. It can easily be verified that vector sum has the property of association, that is,

$$(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C}).$$

## Vector subtraction

Since  $\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B})$ , in order to subtract  $\mathbf{B}$  from  $\mathbf{A}$ , we simply multiply  $\mathbf{B}$  by  $-1$  and then add.

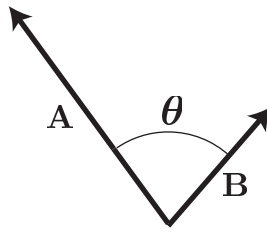


## Scalar product (“Dot” product)

This product involves two vectors and results in a scalar quantity. The scalar product between two vectors  $\mathbf{A}$  and  $\mathbf{B}$ , is denoted by  $\mathbf{A} \cdot \mathbf{B}$ , and is defined as

$$\mathbf{A} \cdot \mathbf{B} = AB \cos \theta .$$

Here  $\theta$ , is the angle between the vectors  $\mathbf{A}$  and  $\mathbf{B}$  when they are drawn with a common origin.



We note that, since  $\cos \theta = \cos(-\theta)$ , it makes no difference which vector is considered first when measuring the angle  $\theta$ . Hence,  $\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$ . If  $\mathbf{A} \cdot \mathbf{B} = 0$ , then either  $A = 0$  and/or  $B = 0$ , or,  $\mathbf{A}$  and  $\mathbf{B}$  are orthogonal, that is,  $\cos \theta = 0$ . We also note that  $\mathbf{A} \cdot \mathbf{A} = A^2$ . If one of the vectors is a unit vector, say  $B = 1$ , then  $\mathbf{A} \cdot \hat{\mathbf{B}} = A \cos \theta$ , is the projection of vector  $\mathbf{A}$  along the direction of  $\hat{\mathbf{B}}$ .

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**Exercise**

Using the definition of scalar product, derive the Law of Cosines which says that, for an arbitrary triangle with sides of length  $A$ ,  $B$ , and  $C$ , we have

$$C^2 = A^2 + B^2 - 2AB \cos \theta .$$

Here,  $\theta$  is the angle opposite side  $C$ . *Hint : associate to each side of the triangle a vector such that  $\mathbf{C} = \mathbf{A} - \mathbf{B}$ , and expand  $C^2 = \mathbf{C} \cdot \mathbf{C}$ .*

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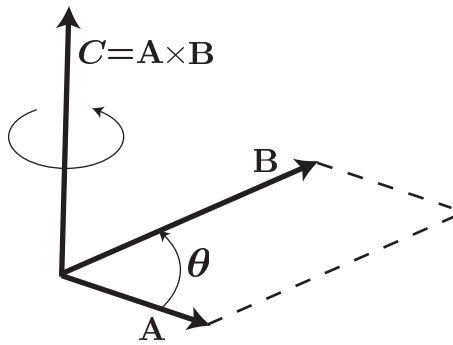
**Vector product (“Cross” product)**

This product operation involves two vectors  $\mathbf{A}$  and  $\mathbf{B}$ , and results in a new vector  $\mathbf{C} = \mathbf{A} \times \mathbf{B}$ . The magnitude of  $\mathbf{C}$  is given by,

$$C = AB \sin \theta ,$$

where  $\theta$  is the angle between the the vectors  $\mathbf{A}$  and  $\mathbf{B}$  when drawn with a common origin. To eliminate ambiguity, between the two possible choices,  $\theta$  is always taken as the angle smaller than  $\pi$ . We can easily show that  $C$  is equal to the area enclosed by the parallelogram defined by  $\mathbf{A}$  and  $\mathbf{B}$ .

The vector  $\mathbf{C}$  is orthogonal to both  $\mathbf{A}$  and  $\mathbf{B}$ , i.e. it is orthogonal to the plane defined by  $\mathbf{A}$  and  $\mathbf{B}$ . The direction of  $\mathbf{C}$  is determined by the *right-hand* rule as shown.



From this definition, it follows that

$$\mathbf{B} \times \mathbf{A} = -\mathbf{A} \times \mathbf{B} ,$$

which indicates that vector multiplication is *not* commutative (but anticommutative). We also note that if  $\mathbf{A} \times \mathbf{B} = \mathbf{0}$ , then, either  $\mathbf{A}$  and/or  $\mathbf{B}$  are zero, or,  $\mathbf{A}$  and  $\mathbf{B}$  are parallel, although not necessarily pointing in the same direction. Thus, we also have  $\mathbf{A} \times \mathbf{A} = \mathbf{0}$ .

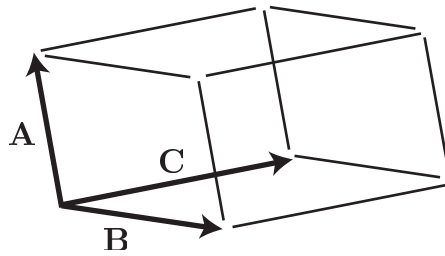
Having defined vector multiplication, it would appear natural to define vector division. In particular, we could say that “ $\mathbf{A}$  divided by  $\mathbf{B}$ ”, is a vector  $\mathbf{C}$  such that  $\mathbf{A} = \mathbf{B} \times \mathbf{C}$ . We see immediately that there are a number of difficulties with this definition. In particular, if  $\mathbf{A}$  is not perpendicular to  $\mathbf{B}$ , the vector  $\mathbf{C}$  does not exist. Moreover, if  $\mathbf{A}$  is perpendicular to  $\mathbf{B}$  then, there are an infinite number of vectors that satisfy  $\mathbf{A} = \mathbf{B} \times \mathbf{C}$ . To see that, let us assume that  $\mathbf{C}$  satisfies,  $\mathbf{A} = \mathbf{B} \times \mathbf{C}$ . Then, any vector  $\mathbf{D} = \mathbf{C} + \beta\mathbf{B}$ , for any scalar  $\beta$ , also satisfies  $\mathbf{A} = \mathbf{B} \times \mathbf{D}$ , since  $\mathbf{B} \times \mathbf{D} = \mathbf{B} \times (\mathbf{C} + \beta\mathbf{B}) = \mathbf{B} \times \mathbf{C} = \mathbf{A}$ . We conclude therefore, that vector division is not a well defined operation.

### ***Exercise***

Show that  $|\mathbf{A} \times \mathbf{B}|$  is the area of the parallelogram defined by the vectors  $\mathbf{A}$  and  $\mathbf{B}$ , when drawn with a common origin.

### **Triple product**

Given three vectors  $\mathbf{A}$ ,  $\mathbf{B}$ , and  $\mathbf{C}$ , the triple product is a scalar given by  $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$ . Geometrically, the triple product can be interpreted as the volume of the three dimensional parallelepiped defined by the three vectors  $\mathbf{A}$ ,  $\mathbf{B}$  and  $\mathbf{C}$ .



It can be easily verified that  $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$ .

### **Exercise**

Show that  $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$  is the volume of the parallelepiped defined by the vectors  $\mathbf{A}$ ,  $\mathbf{B}$ , and  $\mathbf{C}$ , when drawn with a common origin.

## **Double vector product**

The double vector product results from repetition of the cross product operation. A useful identity here is,

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C} .$$

Using this identity we can easily verify that the double cross product is not associative, that is,

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \neq (\mathbf{A} \times \mathbf{B}) \times \mathbf{C} .$$

## **Vector Calculus**

Vector differentiation and integration follow standard rules. Thus if a vector is a function of, say time, then its derivative with respect to time is also a vector. Similarly the integral of a vector is also a vector.

### **Derivative of a vector**

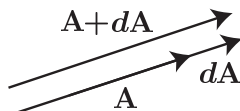
Consider a vector  $\mathbf{A}(t)$  which is a function of, say, time. The derivative of  $\mathbf{A}$  with respect to time is defined as,

$$\frac{d\mathbf{A}}{dt} = \lim_{\Delta t \rightarrow 0} \frac{\mathbf{A}(t + \Delta t) - \mathbf{A}(t)}{\Delta t} . \quad (1)$$

A vector has magnitude and direction, and it changes whenever either of them changes. Therefore the rate of change of a vector will be equal to the sum of the changes due to magnitude and direction.

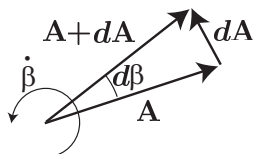
### Rate of change due to magnitude changes

When a vector only changes in magnitude from  $\mathbf{A}$  to  $\mathbf{A} + d\mathbf{A}$ , the rate of change vector  $d\mathbf{A}$  is clearly parallel to the original vector  $\mathbf{A}$ .



### Rate of change due to direction changes

Let us look at the situation where only the direction of the vector changes, while the magnitude stays constant. This is illustrated in the figure where a vector  $\mathbf{A}$  undergoes a small rotation. From the sketch, it is clear that if the magnitude of the vector does not change,  $d\mathbf{A}$  is perpendicular to  $\mathbf{A}$  and as a consequence, the derivative of  $\mathbf{A}$ , must be perpendicular to  $\mathbf{A}$ . (Note that in the picture  $d\mathbf{A}$  has a finite magnitude and therefore,  $\mathbf{A}$  and  $d\mathbf{A}$  are not exactly perpendicular. In reality,  $d\mathbf{A}$  has infinitesimal length and we can see that when the magnitude of  $d\mathbf{A}$  tends to zero,  $\mathbf{A}$  and  $d\mathbf{A}$  are indeed perpendicular).



An alternative, more mathematical, explanation can be derived by realizing that even if  $\mathbf{A}$  changes but its modulus stays constant, then  $\mathbf{A} \cdot \mathbf{A} = 1$ . Differentiating, we have that,

$$d\mathbf{A} \cdot \mathbf{A} + \mathbf{A} \cdot d\mathbf{A} = 2\mathbf{A} \cdot d\mathbf{A} = 0 ,$$

which shows that  $\mathbf{A}$ , and  $d\mathbf{A}$ , must be orthogonal.

Suppose that  $\mathbf{A}$  is instantaneously rotating in the plane of the paper at a rate  $\dot{\beta} = d\beta/dt$ , with no change in magnitude. In an instant  $dt$ ,  $\mathbf{A}$ , will rotate an amount  $d\beta = \dot{\beta}dt$  and the magnitude of  $d\mathbf{A}$ , will be

$$dA = |d\mathbf{A}| = A d\beta = A \dot{\beta} dt .$$

Hence, the magnitude of the vector derivative is

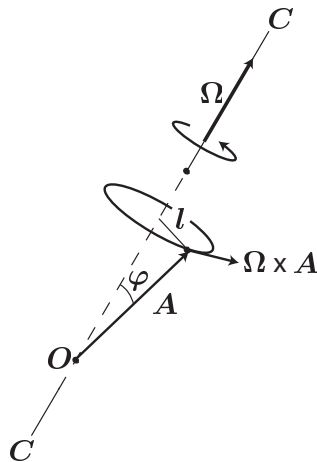
$$\left| \frac{d\mathbf{A}}{dt} \right| = A \dot{\beta} .$$

In the general three dimensional case the situation is a little bit more complicated because the rotation of the vector may occur around a general axis. If we express the instantaneous rotation of  $\mathbf{A}$  in terms of an angular

velocity  $\boldsymbol{\Omega}$  (recall that the angular velocity vector is aligned with the axis of rotation and the direction of the rotation is determined by the right hand rule), then the derivative of  $\mathbf{A}$  with respect to time is simply,

$$\left(\frac{d\mathbf{A}}{dt}\right)_{\text{constant magnitude}} = \boldsymbol{\Omega} \times \mathbf{A} . \quad (2)$$

To see that, consider a vector  $\mathbf{A}$  rotating about the axis  $C - C$  with an angular velocity  $\boldsymbol{\Omega}$ . The derivative will be the velocity of the tip of  $\mathbf{A}$ . Its magnitude is given by  $l\Omega$ , and its direction is both perpendicular to  $\mathbf{A}$  and to the axis of rotation. We note that  $\boldsymbol{\Omega} \times \mathbf{A}$  has the right direction, and the right magnitude since  $l = A \sin \varphi$ .

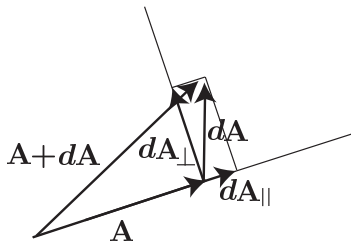


Expression (2) is also valid in the more general case where  $\mathbf{A}$  is rotating about an axis which does not pass through the origin of  $\mathbf{A}$ . We will see in the course, that a rotation about an arbitrary axis can always be written as a rotation about a parallel axis plus a translation, and translations do not affect the magnitude not the direction of a vector.

We can now go back to the general expression for the derivative of a vector (1) and write

$$\frac{d\mathbf{A}}{dt} = \left(\frac{d\mathbf{A}}{dt}\right)_{\text{constant direction}} + \left(\frac{d\mathbf{A}}{dt}\right)_{\text{constant magnitude}} = \left(\frac{d\mathbf{A}}{dt}\right)_{\text{constant direction}} + \boldsymbol{\Omega} \times \mathbf{A} .$$

Note that  $(d\mathbf{A}/dt)_{\text{constant direction}}$  is parallel to  $\mathbf{A}$  and  $\boldsymbol{\Omega} \times \mathbf{A}$  is orthogonal to  $\mathbf{A}$ . The figure below shows the general differential of a vector, which has a component which is parallel to  $\mathbf{A}$ ,  $d\mathbf{A}_{\parallel}$ , and a component which is orthogonal to  $\mathbf{A}$ ,  $d\mathbf{A}_{\perp}$ . The magnitude change is given by  $dA_{\parallel}$ , and the direction change is given by  $d\mathbf{A}_{\perp}$ .



## Rules for Vector Differentiation

Vector differentiation follows similar rules to scalars regarding vector addition, multiplication by a scalar, and products. In particular we have that, for any vectors  $\mathbf{A}$ ,  $\mathbf{B}$ , and any scalar  $\alpha$ ,

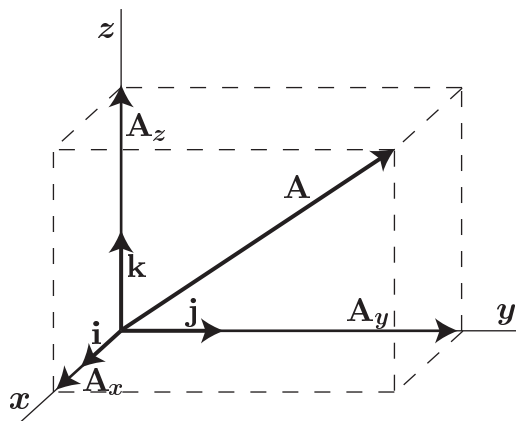
$$\begin{aligned}d(\alpha\mathbf{A}) &= d\alpha\mathbf{A} + \alpha d\mathbf{A} \\d(\mathbf{A} + \mathbf{B}) &= d\mathbf{A} + d\mathbf{B} \\d(\mathbf{A} \cdot \mathbf{B}) &= d\mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot d\mathbf{B} \\d(\mathbf{A} \times \mathbf{B}) &= d\mathbf{A} \times \mathbf{B} + \mathbf{A} \times d\mathbf{B} .\end{aligned}$$

## Components of a Vector

We have seen above that it is possible to define several operations involving vectors without ever introducing a reference frame. This is a rather important concept which explains why vectors and vector equations are so useful to express physical laws, since these, must be obviously independent of any particular frame of reference.

In practice however, reference frames need to be introduced at some point in order to express, or measure, the direction and magnitude of vectors, i.e. we can easily measure the direction of a vector by measuring the angle that the vector makes with the local vertical and the geographic north.

Consider a *right-handed* set of axes  $xyz$ , defined by three mutually orthogonal unit vectors  $\mathbf{i}$ ,  $\mathbf{j}$  and  $\mathbf{k}$  ( $\mathbf{i} \times \mathbf{j} = \mathbf{k}$ ) (note that here we are not using the hat (^) notation). Since the vectors  $\mathbf{i}$ ,  $\mathbf{j}$  and  $\mathbf{k}$  are mutually orthogonal they form a basis. The projections of  $\mathbf{A}$  along the three  $xyz$  axes are the *components* of  $\mathbf{A}$  in the  $xyz$  reference frame.



In order to determine the components of  $\mathbf{A}$ , we can use the scalar product and write,

$$A_x = \mathbf{A} \cdot \mathbf{i}, \quad A_y = \mathbf{A} \cdot \mathbf{j}, \quad A_z = \mathbf{A} \cdot \mathbf{k} .$$

The vector  $\mathbf{A}$ , can thus be written as a sum of the three vectors along the coordinate axis which have

magnitudes  $A_x$ ,  $A_y$ , and  $A_z$ .

$$\mathbf{A} = \mathbf{A}_x + \mathbf{A}_y + \mathbf{A}_z = A_x \mathbf{i} + A_y \mathbf{j} + A_z \mathbf{k} = (\mathbf{i}, \mathbf{j}, \mathbf{k}) \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}.$$

## Vector operations in component form

The vector operations introduced above can be expressed in terms of the vector components in a rather straightforward manner. For instance, when we say that  $\mathbf{A} = \mathbf{B}$ , this implies that the projections of  $\mathbf{A}$  and  $\mathbf{B}$  along the  $xyz$  axes are the same, and therefore, this is equivalent to three scalar equations e.g.  $A_x = B_x$ ,  $A_y = B_y$ , and  $A_z = B_z$ . Regarding vector summation, subtraction and multiplication by a scalar, we have that, if  $\mathbf{C} = \alpha \mathbf{A} + \beta \mathbf{B}$ , then,

$$C_x = \alpha A_x + \beta B_x, \quad C_y = \alpha A_y + \beta B_y, \quad C_z = \alpha A_z + \beta B_z.$$

## Scalar product

Since  $\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1$  and that  $\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0$ , the scalar product of two vectors can be written as,

$$\mathbf{A} \cdot \mathbf{B} = (A_x \mathbf{i} + A_y \mathbf{j} + A_z \mathbf{k}) \cdot (B_x \mathbf{i} + B_y \mathbf{j} + B_z \mathbf{k}) = A_x B_x + A_y B_y + A_z B_z.$$

Note that,  $\mathbf{A} \cdot \mathbf{A} = A^2 = A_x^2 + A_y^2 + A_z^2$ , which is consistent with Pythagoras' theorem.

## Vector product

Here,  $\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}$  and  $\mathbf{i} \times \mathbf{j} = \mathbf{k}$ ,  $\mathbf{j} \times \mathbf{k} = \mathbf{i}$ , and  $\mathbf{k} \times \mathbf{i} = \mathbf{j}$ . Thus,

$$\begin{aligned} \mathbf{A} \times \mathbf{B} &= (A_x \mathbf{i} + A_y \mathbf{j} + A_z \mathbf{k}) \times (B_x \mathbf{i} + B_y \mathbf{j} + B_z \mathbf{k}) \\ &= (A_y B_z - A_z B_y) \mathbf{i} + (A_z B_x - A_x B_z) \mathbf{j} + (A_x B_y - A_y B_x) \mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}. \end{aligned}$$

## Triple product

The triple product  $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$  can be expressed as the following determinant

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix},$$

which clearly is equal to zero whenever the vectors are linearly dependent (if the three vectors are linearly dependent they must be co-planar and therefore the parallelipiped defined by the three vectors has zero volume).

## Change of basis

In many problems we will need to use different basis in order to describe different vector quantities. The above operations, written in component form, only make sense once all the vectors involved are described with respect to the same frame. In this section, we will see how the components of a vector are transformed when we change the reference frame.

Consider two different orthogonal, right-hand sided, reference frames  $xyz$  and  $x'y'z'$ , with associated basis  $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ , and  $(\mathbf{i}', \mathbf{j}', \mathbf{k}')$ . First, we express the vectors  $\mathbf{i}$ ,  $\mathbf{j}$  and  $\mathbf{k}$  with respect to the basis  $(\mathbf{i}', \mathbf{j}', \mathbf{k}')$ . To this end, we write for  $\mathbf{i}$ ,

$$\mathbf{i} = i_x \mathbf{i}' + i_y \mathbf{j}' + i_z \mathbf{k}' ,$$

where,  $i_x$ ,  $i_y$  and  $i_z$  are the components of  $\mathbf{i}$  with respect to  $(\mathbf{i}', \mathbf{j}', \mathbf{k}')$ . Since these components are the projections of  $\mathbf{i}$  along the  $x'y'z'$  axes we have,

$$i_x = \mathbf{i} \cdot \mathbf{i}' , \quad i_y = \mathbf{i} \cdot \mathbf{j}' , \quad i_z = \mathbf{i} \cdot \mathbf{k}' ,$$

or

$$\mathbf{i} = (\mathbf{i} \cdot \mathbf{i}') \mathbf{i}' + (\mathbf{i} \cdot \mathbf{j}') \mathbf{j}' + (\mathbf{i} \cdot \mathbf{k}') \mathbf{k}' .$$

Analogously, for  $\mathbf{j}$  and  $\mathbf{k}$ , we can write,

$$\mathbf{j} = (\mathbf{j} \cdot \mathbf{i}') \mathbf{i}' + (\mathbf{j} \cdot \mathbf{j}') \mathbf{j}' + (\mathbf{j} \cdot \mathbf{k}') \mathbf{k}' , \quad \mathbf{k} = (\mathbf{k} \cdot \mathbf{i}') \mathbf{i}' + (\mathbf{k} \cdot \mathbf{j}') \mathbf{j}' + (\mathbf{k} \cdot \mathbf{k}') \mathbf{k}' .$$

The above relations can be written in matrix form as

$$(\mathbf{i}, \mathbf{j}, \mathbf{k}) = (\mathbf{i}', \mathbf{j}', \mathbf{k}') \begin{pmatrix} \mathbf{i} \cdot \mathbf{i}' & \mathbf{j} \cdot \mathbf{i}' & \mathbf{k} \cdot \mathbf{i}' \\ \mathbf{i} \cdot \mathbf{j}' & \mathbf{j} \cdot \mathbf{j}' & \mathbf{k} \cdot \mathbf{j}' \\ \mathbf{i} \cdot \mathbf{k}' & \mathbf{j} \cdot \mathbf{k}' & \mathbf{k} \cdot \mathbf{k}' \end{pmatrix} . \quad (3)$$

The vector  $\mathbf{A}$ , can be written as

$$\mathbf{A} = A_x \mathbf{i} + A_y \mathbf{j} + A_z \mathbf{k} = (\mathbf{i}, \mathbf{j}, \mathbf{k}) \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} , \quad (4)$$

or, when referred to the frame  $x'y'z'$ , as

$$\mathbf{A} = A'_x \mathbf{i}' + A'_y \mathbf{j}' + A'_z \mathbf{k}' = (\mathbf{i}', \mathbf{j}', \mathbf{k}') \begin{pmatrix} A'_x \\ A'_y \\ A'_z \end{pmatrix} . \quad (5)$$

Inserting equation (3) into equation (4), and equating expressions (4) and (5), we obtain,

$$\mathbf{A} = (\mathbf{i}', \mathbf{j}', \mathbf{k}') \begin{pmatrix} \mathbf{i} \cdot \mathbf{i}' & \mathbf{j} \cdot \mathbf{i}' & \mathbf{k} \cdot \mathbf{i}' \\ \mathbf{i} \cdot \mathbf{j}' & \mathbf{j} \cdot \mathbf{j}' & \mathbf{k} \cdot \mathbf{j}' \\ \mathbf{i} \cdot \mathbf{k}' & \mathbf{j} \cdot \mathbf{k}' & \mathbf{k} \cdot \mathbf{k}' \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = (\mathbf{i}', \mathbf{j}', \mathbf{k}') \begin{pmatrix} A'_x \\ A'_y \\ A'_z \end{pmatrix} .$$

Since  $(\mathbf{i}', \mathbf{j}', \mathbf{k}')$  are independent, this implies that,

$$\begin{pmatrix} A'_x \\ A'_y \\ A'_z \end{pmatrix} = \begin{pmatrix} \mathbf{i} \cdot \mathbf{i}' & \mathbf{j} \cdot \mathbf{i}' & \mathbf{k} \cdot \mathbf{i}' \\ \mathbf{i} \cdot \mathbf{j}' & \mathbf{j} \cdot \mathbf{j}' & \mathbf{k} \cdot \mathbf{j}' \\ \mathbf{i} \cdot \mathbf{k}' & \mathbf{j} \cdot \mathbf{k}' & \mathbf{k} \cdot \mathbf{k}' \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = [T] \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}.$$

The above expression is the relationship that expresses how the components of a vector in one basis relate to the components of the *same* vector in a different basis.

The components of the transformation, or rotation, matrix  $[T]$  are nothing but the cosines of the angles that the vectors of the new basis  $(\mathbf{i}', \mathbf{j}', \mathbf{k}')$ , form with the vectors of the original basis  $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ . They are often referred to as the *direction cosines*. The transformation matrix is an orthogonal matrix, which has the interesting property that its inverse is equal to its transpose. In addition, the determinant of the rotation matrix can be shown to be always equal to one.

The fact that the inverse of the transformation matrix is equal to its transpose can be seen by reversing the roles of  $xyz$  and  $x'y'z'$  in the derivation of  $[T]$ , or in the same way, reversing the roles of  $(\mathbf{i}, \mathbf{j}, \mathbf{k})$  and  $(\mathbf{i}', \mathbf{j}', \mathbf{k}')$ . In this case we obtain,

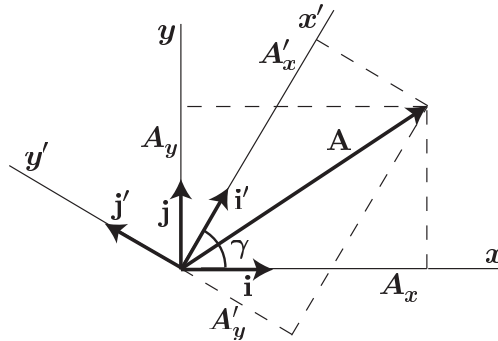
$$\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} \mathbf{i} \cdot \mathbf{i}' & \mathbf{i} \cdot \mathbf{j}' & \mathbf{i} \cdot \mathbf{k}' \\ \mathbf{j} \cdot \mathbf{i}' & \mathbf{j} \cdot \mathbf{j}' & \mathbf{j} \cdot \mathbf{k}' \\ \mathbf{k} \cdot \mathbf{i}' & \mathbf{k} \cdot \mathbf{j}' & \mathbf{k} \cdot \mathbf{k}' \end{pmatrix} \begin{pmatrix} A'_x \\ A'_y \\ A'_z \end{pmatrix} = [T]^{-1} \begin{pmatrix} A'_x \\ A'_y \\ A'_z \end{pmatrix},$$

which clearly shows that  $[T]^{-1} = [T]^T$ .

### Example

### Change of basis in two dimensions

Here, we apply for illustration purposes, the above expressions to a two dimensional example. Consider the change of coordinates between two reference frames  $xy$ , and  $x'y'$ , as shown in the diagram.



The angle between  $\mathbf{i}$  and  $\mathbf{i}'$  is  $\gamma$ . Therefore,  $\mathbf{i} \cdot \mathbf{i}' = \cos \gamma$ . Similarly,  $\mathbf{j} \cdot \mathbf{i}' = \cos(\pi/2 - \gamma) = \sin \gamma$ ,  $\mathbf{i} \cdot \mathbf{j}' = \cos(\pi/2 + \gamma) = -\sin \gamma$ , and  $\mathbf{j} \cdot \mathbf{j}' = \cos \gamma$ . Finally, the transformation matrix  $[T]$  is

$$[T] = \begin{pmatrix} \mathbf{i} \cdot \mathbf{i}' & \mathbf{j} \cdot \mathbf{i}' \\ \mathbf{i} \cdot \mathbf{j}' & \mathbf{j} \cdot \mathbf{j}' \end{pmatrix} = \begin{pmatrix} \cos \gamma & \sin \gamma \\ -\sin \gamma & \cos \gamma \end{pmatrix},$$

and we can write,

$$\begin{pmatrix} A'_x \\ A'_y \end{pmatrix} = [T] \begin{pmatrix} A_x \\ A_y \end{pmatrix}, \quad \begin{pmatrix} A_x \\ A_y \end{pmatrix} = [T]^T \begin{pmatrix} A'_x \\ A'_y \end{pmatrix}.$$

For instance, we can easily check that when  $\gamma = \pi/2$ , the above expressions give  $A'_x = A_y$ , and  $A'_y = -A_x$ , as expected.

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## References

- [1] D. Kleppner and R.J. Kolenkow, *An Introduction to Mechanics*, McGraw Hill, 1973.