

Unit 13

Review of Simple Beam Theory

Readings:

Review Unified Engineering notes on Beam Theory

BMP 3.8, 3.9, 3.10

T & G 120-125

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IV. General Beam Theory

We have thus far looked at:

- in-plane loads
- torsional loads

In addition, structures can carry loads by *bending*. The 2-D case is a *plate*, the simple 1-D case is a *beam*. Let's first review what you learned in Unified as **Simple Beam Theory**

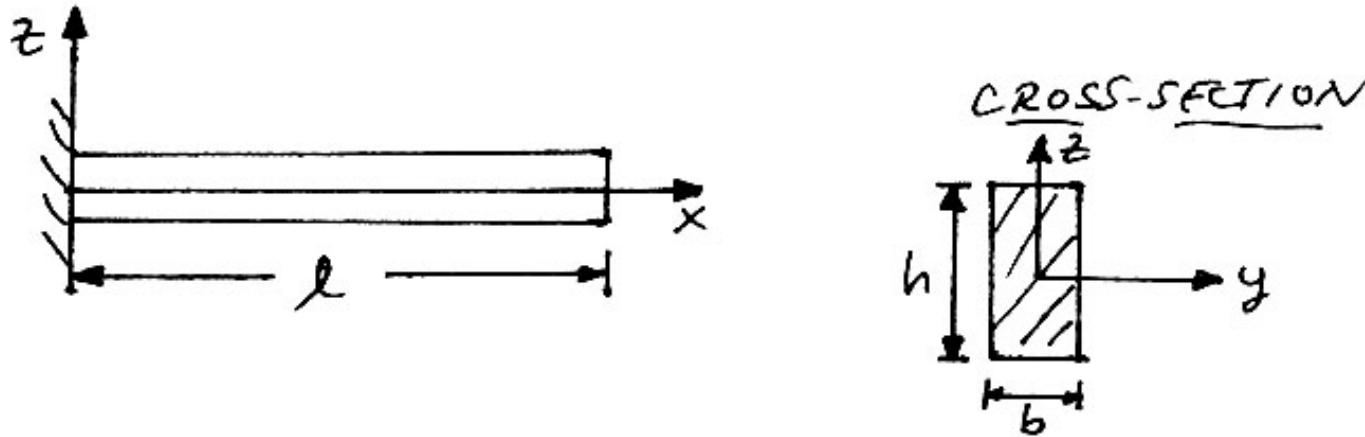
(review of) Simple Beam Theory

A *beam* is a bar capable of carrying loads in bending. The loads are applied transverse to its longest dimension.

Assumptions:

1. Geometry

Figure 13.1 General Geometry of a Beam



- long & thin $\Rightarrow l \gg b, h$
- loading is in z -direction
- loading passes through “shear center” \Rightarrow no torsion/twist
(we’ll define this term later and relax this constraint.)
- cross-section can vary along x

2. Stress state

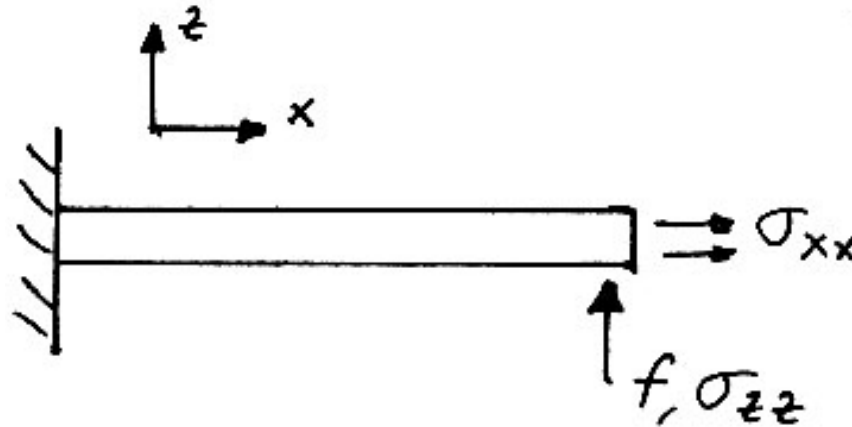
- $\sigma_{yy}, \sigma_{yz}, \sigma_{xy} = 0 \Rightarrow$ no stress in y -direction
 - $\sigma_{xx} \gg \sigma_{zz}$
- $\sigma_{xz} \gg \sigma_{zz} \Rightarrow$ only significant stresses are σ_{xx} and σ_{xz}

Note: there is a load in the z-direction to cause these stresses, but generated σ_{xx} is much larger (similar to pressurized cylinder example)

Why is this valid?

Look at moment arms:

Figure 13.2 Representation of force applied in beam



σ_{xx} moment arm is order of (h)

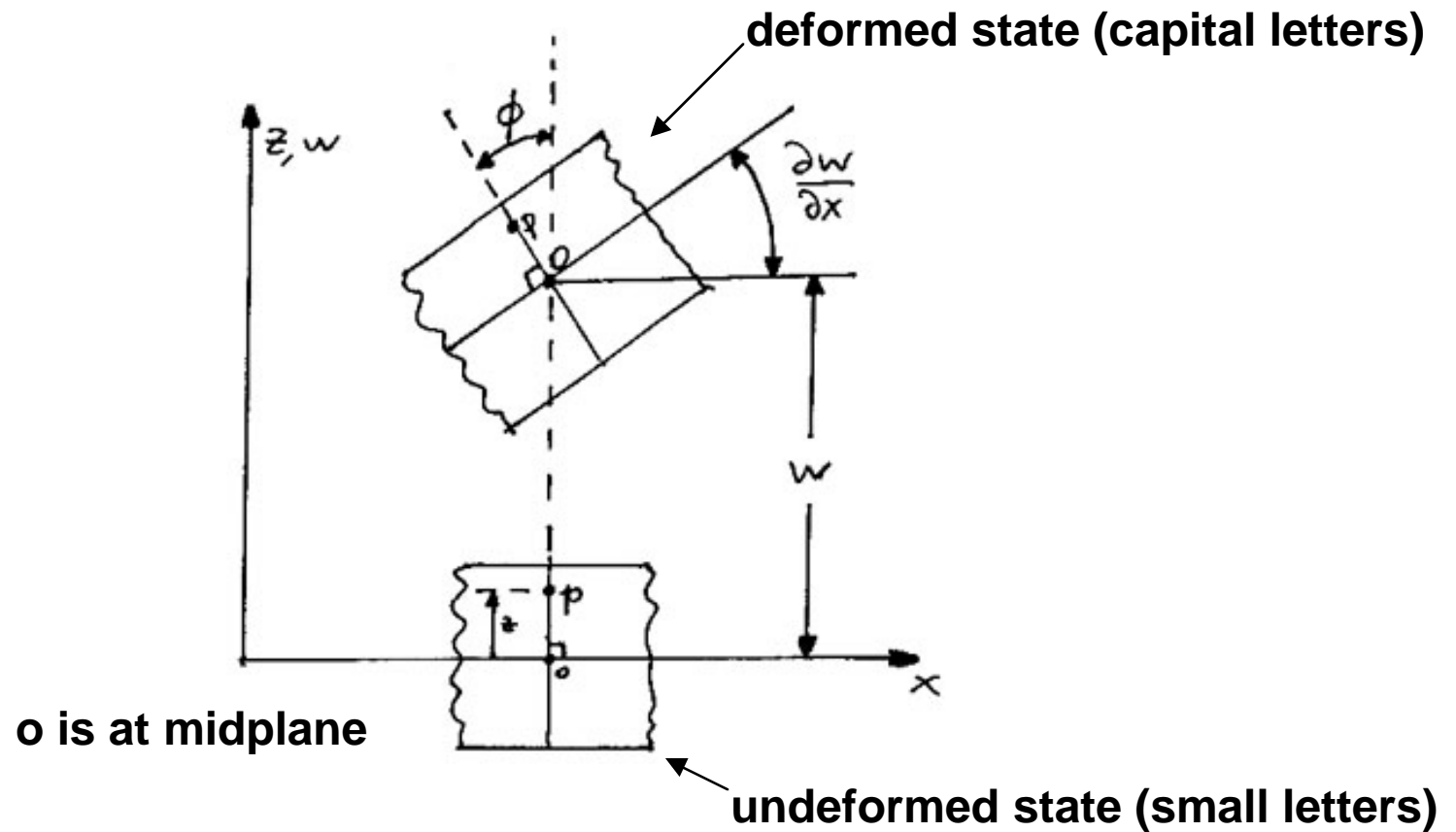
σ_{zz} moment arm is order of (l)

and $l \gg h$

$\Rightarrow \sigma_{xx} \gg \sigma_{zz}$ for equilibrium

3. Deformation

Figure 13.3 Representation of deformation of cross-section of a beam



define: w = deflection of midplane (function of x only)

- a) Assume plane sections remain plane and perpendicular to the midplane after deformation

“Bernoulli - Euler Hypothesis” ~ 1750

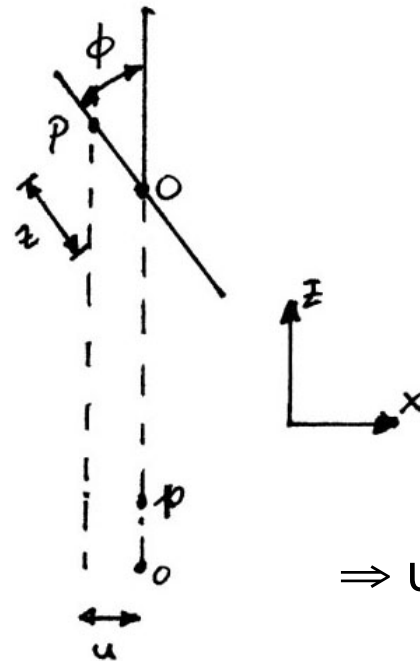
- b) For small angles, this implies the following for deflections:

$$u(x, y, z) \approx -z\phi \approx -z \frac{dw}{dx} \quad (13 - 1)$$

$$\left(\phi = \frac{dw}{dx} \right)$$

total derivative
since it does not
vary with y or z

Figure 13.4 Representation of movement in x-direction of two points on same plane in beam



$$\Rightarrow u = -z \sin \phi \quad \text{Note direction of } u \text{ relative to } +x \text{ direction}$$

and for ϕ small:

$$\Rightarrow u = -z \phi$$

$$v(x, y, z) = 0$$

$$w(x, y, z) \approx w(x) \quad (13 - 2)$$

Now look at the strain-displacement equations:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = -z \frac{d^2 w}{dx^2} \quad (13 - 3)$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} = 0$$

$$\varepsilon_{zz} = \frac{\partial w}{\partial z} = 0 \quad (\text{no deformation through thickness})$$

$$\varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0$$

$$\varepsilon_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = 0$$

$$\varepsilon_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = -\frac{\partial w}{\partial x} + \frac{\partial w}{\partial x} = 0$$

Now consider the stress-strain equations (for the time being consider isotropic...extend this to orthotropic later)

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E} \quad (13 - 4)$$

$$\varepsilon_{yy} = -\frac{\nu\sigma_{xx}}{E} \quad \leftarrow \text{small inconsistency with previous}$$

$$\varepsilon_{zz} = -\frac{\nu\sigma_{xx}}{E} \quad \leftarrow \text{small inconsistency with previous}$$

$$\varepsilon_{xy} = \frac{2(1+\nu)}{E}\sigma_{xy} = 0$$

$$\varepsilon_{yz} = \frac{2(1+\nu)}{E}\sigma_{yz} = 0$$

$$\varepsilon_{zx} = \frac{2(1+\nu)}{E}\sigma_{zx} \quad \leftarrow \text{inconsistency again!}$$

We get around these inconsistencies by saying that ε_{yy} , ε_{zz} , ε_{xz} are very small but not quite zero. This is an **approximation**. We will evaluate these later on.

4. Equilibrium Equations

Assumptions:

- a) no body forces
- b) equilibrium in y-direction is “ignored”
- c) x, z equilibrium are satisfied in an average sense

So:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xz}}{\partial z} = 0 \quad (13 - 5)$$

$$0 = 0 \quad (\text{y -equilibrium})$$

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{zz}}{\partial z} = 0 \quad (13 - 6)$$

Note, average equilibrium equations:

$$\iint_{\text{face}} [\text{Eq. (13 - 6)}] dy dz \Rightarrow \frac{dS}{dx} = p \quad (13 - 6a)$$

$$\iint_{\text{face}} z [\text{Eq. (13 - 5)}] dy dz \Rightarrow \frac{dM}{dx} = S \quad (13 - 5a)$$

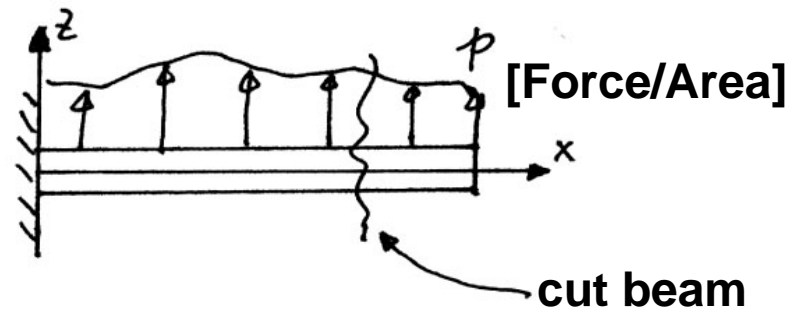
These are the Moment, Shear, Loading relations where the stress resultants are:

$$\text{Axial Force} \quad F = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{xx} b \, dz \quad (13 - 7)$$

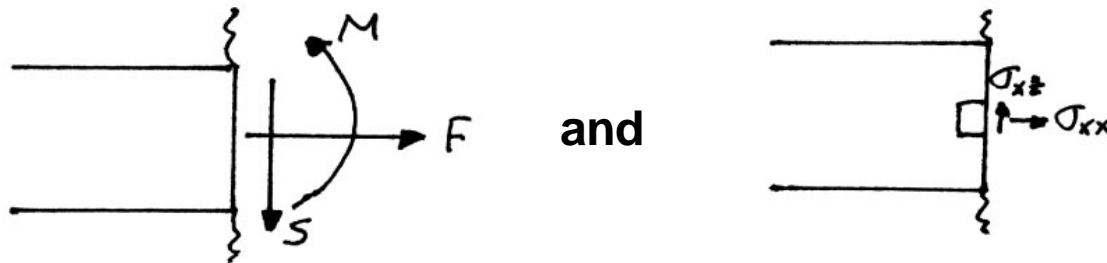
$$\text{Shear Force} \quad S = - \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{xz} b \, dz \quad (13 - 8)$$

$$\text{Bending Moment} \quad M = - \int_{-\frac{h}{2}}^{\frac{h}{2}} z \sigma_{xx} b \, dz \quad (13 - 9)$$

Figure 13.8 Representation of Moment, Shear and Loading on a beam



(F, S, M found from statics)



So the final important equations of Simple Beam Theory are:

$$\varepsilon_{xx} = -z \frac{d^2 w}{dx^2} = \frac{\partial u}{\partial x} \quad (13 - 3)$$

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E} \quad (13 - 4)$$

$$\frac{dS}{dx} = p \quad (13 - 6a)$$

$$\frac{dM}{dx} = S \quad (13 - 5a)$$

--> How do these change if the material is orthotropic?

We have assumed that the properties along x dominate and have ignored ε_{yy} , etc.

Thus, use E_L in the above equations.

But, approximation may not be as good since

ε_{yy} , ε_{zz} , ε_{xz} may be large and really not close enough to zero to be assumed approximately equal to zero

Solution of Equations

using (13 - 3) and (13 - 4) we get:

$$\sigma_{xx} = E \varepsilon_{xx} = -Ez \frac{d^2 w}{dx^2} \quad (13 - 10)$$

Now use this in the expression for the axial force of equation (13 - 7):

$$\begin{aligned} F &= -E \frac{d^2 w}{dx^2} \int_{-\frac{h}{2}}^{\frac{h}{2}} z b dz \\ &= -E \frac{d^2 w}{dx^2} \left[\frac{z^2}{2} b \right]_{-\frac{h}{2}}^{\frac{h}{2}} = 0 \end{aligned}$$

No axial force in beam theory

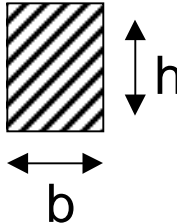
(Note: something that carries axial and bending forces is known as a *beam-column*)

Now place the stress expression (13 - 10) into the moment equation (13 - 9):

$$M = E \frac{d^2 w}{dx^2} \int_{-\frac{h}{2}}^{\frac{h}{2}} z^2 b dz$$

definition: $I = \int_{-\frac{h}{2}}^{\frac{h}{2}} z^2 b dz$ moment of inertia of cross-section

for rectangular cross-section:

$$I = \frac{bh^3}{12} \quad [\text{length}^4]$$


Thus:

$$\boxed{M = EI \frac{d^2 w}{dx^2}} \quad (13 - 11)$$

“Moment - Curvature Relation”

--> Now place equation (13 - 11) into equation (13 - 10) to arrive at:

$$\sigma_{xx} = -Ez \frac{M}{EI}$$

$$\Rightarrow \boxed{\sigma_{xx} = -\frac{Mz}{I}} \quad (13 - 12)$$

Finally, we can get an expression for the shear stress by using equation (13 - 5):

$$\frac{\partial \sigma_{xz}}{\partial z} = - \frac{\partial \sigma_{xx}}{\partial x} \quad (13 - 5)$$

Multiply this by b and integrate from z to $h/2$ to get:

$$\int_z^{\frac{h}{2}} b \frac{\partial \sigma_{xz}}{\partial z} dz = - \int_z^{\frac{h}{2}} \frac{\partial \sigma_{xx}}{\partial x} b dz$$

$$\Rightarrow b \left[\underbrace{\sigma_{xz}(h/2)}_{=0} - \sigma_{xz}(z) \right] = - \int_z^{\frac{h}{2}} \underbrace{\frac{\partial}{\partial x} \left(-\frac{Mz}{I} \right)}_{= -\frac{z}{I} \frac{dM}{dx}} b dz$$

(from boundary condition
of no stress on top surface)

$$= S$$

$$(13 - 13)$$

This all gives:

$$\Rightarrow \boxed{\sigma_{xz} = -\frac{SQ}{Ib}}$$

where:

$$Q = \int_z^{\frac{h}{2}} z b dz \quad = \text{Moment of the area above the center}$$

function of z -- maximum occurs at $z = 0$

Summarizing:

$$\frac{dS}{dx} = p$$

$$\frac{dM}{dx} = S$$

$$\sigma_{xx} = -\frac{Mz}{I}$$

$$\sigma_{xz} = -\frac{SQ}{Ib}$$

$$M = EI \frac{d^2w}{dx^2}$$

Notes:

- σ_{xx} is linear through thickness and zero at midpoint
- σ_{xz} has parabolic distribution through thickness with maximum at midpoint
- Usually $\sigma_{xx} \gg \sigma_{zz}$

Solution Procedure

1. Draw free body diagram
2. Calculate reactions
3. Obtain shear via (13 - 6a) and then σ_{xz} via (13 - 13)
4. Obtain moment via (13 - 5a) and then σ_{xx} via (13 - 12) and deflection via (13 - 11)

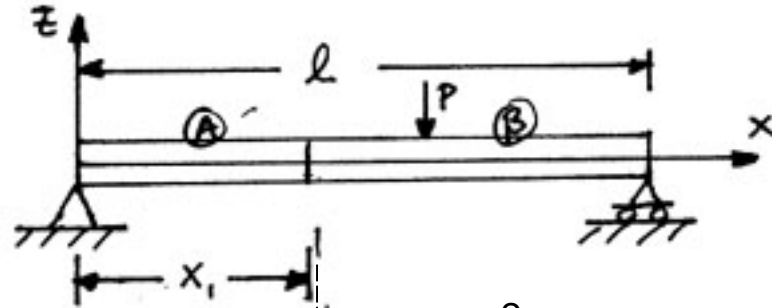
NOTE: steps 2 through 4 must be solved simultaneously if loading is indeterminate

Notes:

- Same formulation for orthotropic material except
 - Use E_L
 - Assumptions on $\varepsilon_{\alpha\beta}$ may get worse
- Can also be solved via stress function approach

- For beams with discontinuities, can solve in each section separately and join (match boundary conditions)

Figure 13.6 Example of solution approach for beam with discontinuity



$$E I \frac{d^2 w_A}{dx^2} = M_A$$

$$E I \frac{d^2 w_B}{dx^2} = M_B$$

$$E I w_A = \dots + C_1 x + C_2$$

$$E I w_B = \dots + C_3 x + C_4$$

--> Subject to Boundary Conditions:

$$@ x = 0, w = w_A = 0$$

$$@ x = x_1, \left\{ \begin{array}{l} w_A = w_B \\ \frac{dw_A}{dx} = \frac{dw_B}{dx} \end{array} \right\} \text{ displacements and slopes match}$$

$$@ x = l, w = w_B = 0$$