Unit 11
Membrane Analogy (for Torsion)

Readings:

Rivello	8.3, 8.6
T & G	107, 108, 109, 110, 112, 113, 114

Paul A. Lagace, Ph.D.
Professor of Aeronautics & Astronautics
and Engineering Systems

For a number of cross-sections, we cannot find stress functions. However, we can resort to an analogy introduced by Prandtl (1903).

Consider a <u>membrane</u> under pressure p_i

"<u>Membrane</u>": structure whose thickness is small compared to surface dimensions and it (thus) has negligible bending rigidity (e.g. soap bubble)

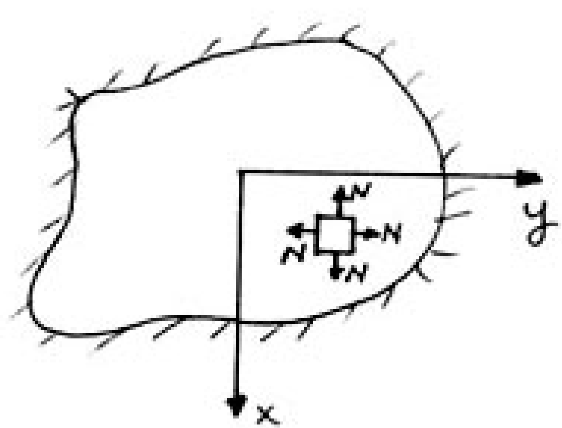
⇒ membrane carries load via a constant tensile force along itself.

N.B. Membrane is 2-D analogy of a string
(plate is 2-D analogy of a beam)

Stretch the membrane over a cutout of the cross-sectional shape in the x-y plane:

Figure 11.1 **Top view of membrane under pressure over cutout**

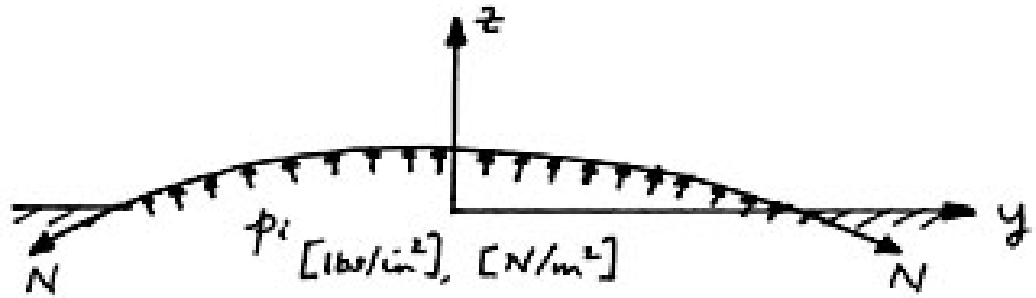
membrane covering a cu



N = constant tension force per unit length [lbs/in] [N/M]

Look at this from the side:

Figure 11.2 **Side view of membrane under pressure over cutout**

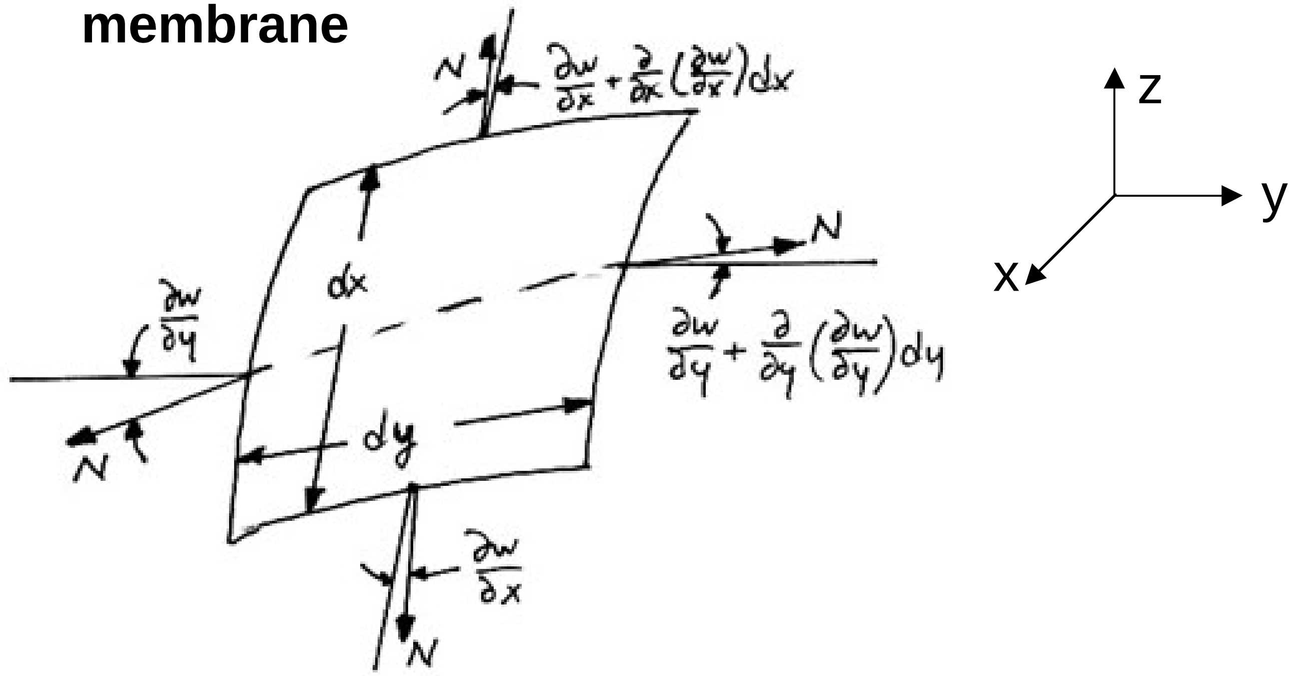


<u>Assume</u>: lateral displacements (w) are small such that no appreciable changes in N occur.

We want to take equilibrium of a small element:

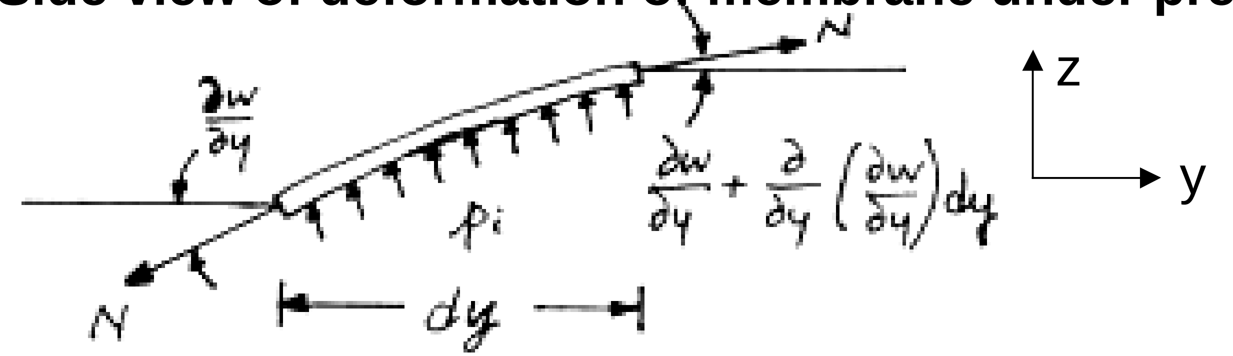
(assume small angles $\frac{\partial w}{\partial x}$, $\frac{\partial w}{\partial y}$)

Figure 11.3 **Representation of deformation of infinitesimal element of membrane**



Look at side view (one side):

Figure 11.4 **Side view of deformation of membrane under pressure**

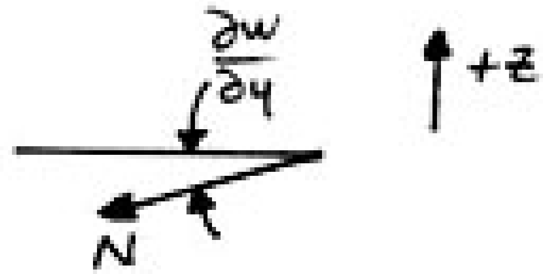


<u>Note</u>: we have similar picture in the x-z plane

We look at equilibrium in the z direction.

Take the z-components of N:

e. g.



z-component = $-N \sin \frac{\partial w}{\partial y}$

note +z direction

for small angle:

$$\sin \frac{\partial w}{\partial y} \approx \frac{\partial w}{\partial y}$$

$\Rightarrow$ z-component $= -N \frac{\partial w}{\partial y}$ (acts over dx face)

With this established, we get:

$$\uparrow + \quad \sum F_z = 0 \Rightarrow \quad p_i \, dxdy \; - \; N \frac{\partial w}{\partial y} dx \; + \; N \left[\frac{\partial w}{\partial y} + \frac{\partial^2 w}{\partial y^2} dy \right] dx$$

$$- \, N \frac{\partial w}{\partial x} dy \; + \; N \left[\frac{\partial w}{\partial x} + \frac{\partial^2 w}{\partial x^2} dx \right] dy \; = \; 0$$

Eliminating like terms and canceling out dxdy gives:

$$p_i \; + \; N \frac{\partial^2 w}{\partial y^2} \; + \; N \frac{\partial^2 w}{\partial x^2} \; = \; 0$$

$$\Rightarrow \boxed{\frac{\partial^2 w}{\partial x^2} \; + \; \frac{\partial^2 w}{\partial y^2} \; = \; - \frac{p_i}{N}}$$

Governing Partial Differential Equation for deflection, w, of a membrane

<u>Boundary</u> <u>Condition</u>: membrane is attached at boundary, so w = 0 along contour

$\Rightarrow$ Exactly the same as torsion problem:

	Torsion	Membrane
Partial Differential Equation	$\nabla^2 \phi = 2Gk$	$\nabla^2 w = -p_i / N$
Boundary Condition	$\phi = 0$ on contour	$w = 0$ on contour

Analogy:

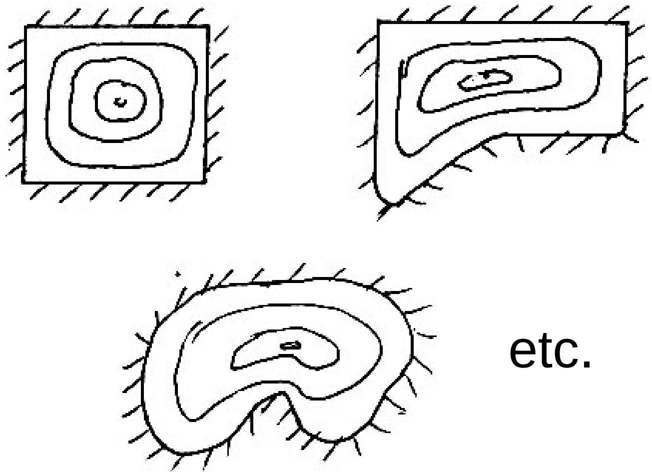
Membrane		Torsion
w	$\rightarrow$	ϕ
p_i	$\rightarrow$	$-k$
N	$\rightarrow$	$\frac{1}{2G}$
$\frac{\partial w}{\partial x}$	$\rightarrow$	$\frac{\partial \phi}{\partial x} = \sigma_{zy}$
$\frac{\partial w}{\partial y}$	$\rightarrow$	$\frac{\partial \phi}{\partial y} = -\sigma_{zx}$
Volume $= \iint w\,dx\,dy$	$\rightarrow$	$-\frac{T}{2}$

Note: for orthotropic, would need a membrane to give different N's in different directions in proportion to G_{xz} and G_{yz}

$\Rightarrow$ Membrane analogy only applies to isotropic materials

- This analogy gives a good "physical" picture for ϕ
- Easy to visualize deflections of membrane for odd shapes

Figure 11.5 **Representation of ϕ and thus deformations for various closed cross-sections under torsion**



etc.

Can use (and people have used) elaborate soap film equipment and measuring devices

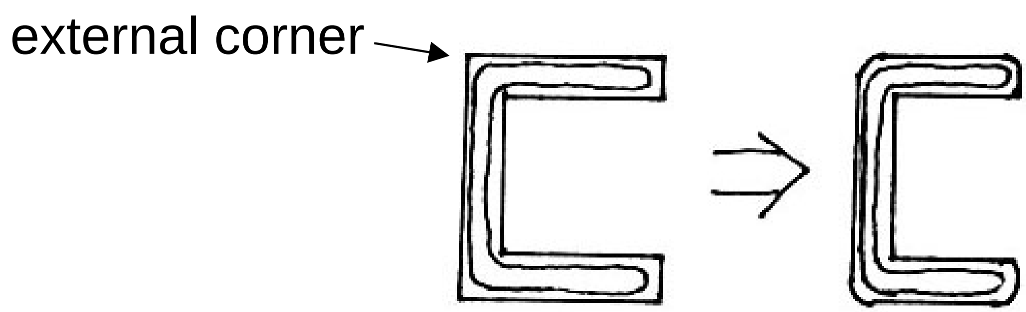
(See Timoshenko, Ch. 11)

From this, can see a number of things:

- Location of maximum shear stresses (at the maximum slopes of the membrane)
- Torque applied (volume of membrane)
- "External" corners do not add appreciability to the bending rigidity (J)

$\Rightarrow$ eliminate these:

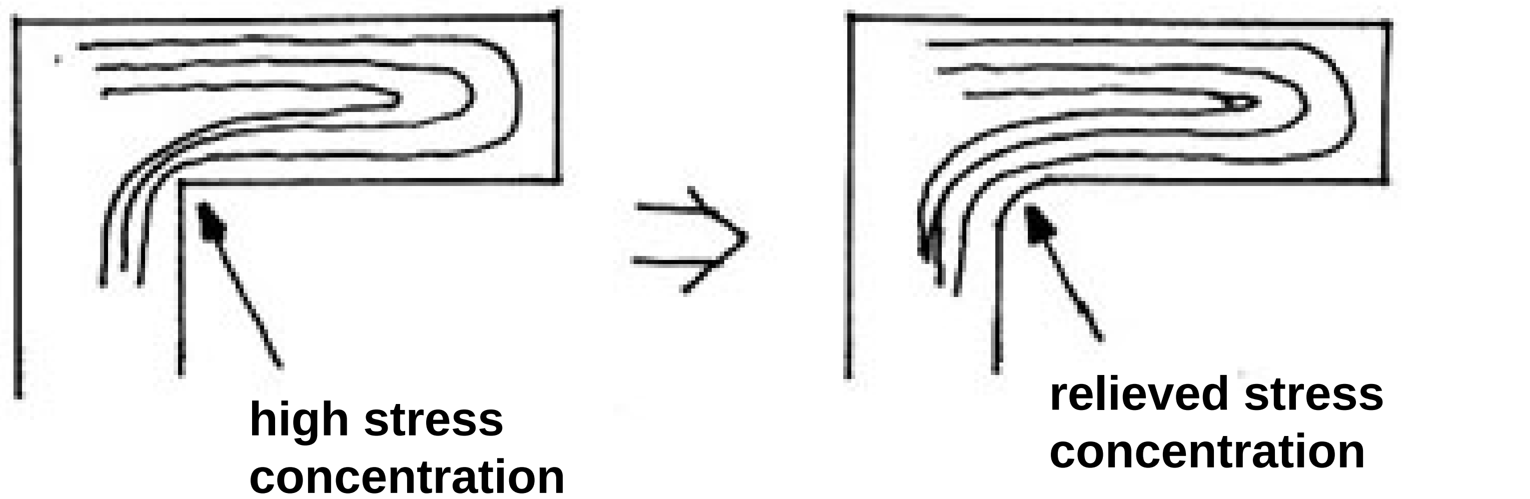
Figure 11.6€ **Representation of effect of external corners**



$\Rightarrow$ **<u>about the same</u>**

- Fillets (i.e. @ internal corners) eliminate stress concentrations

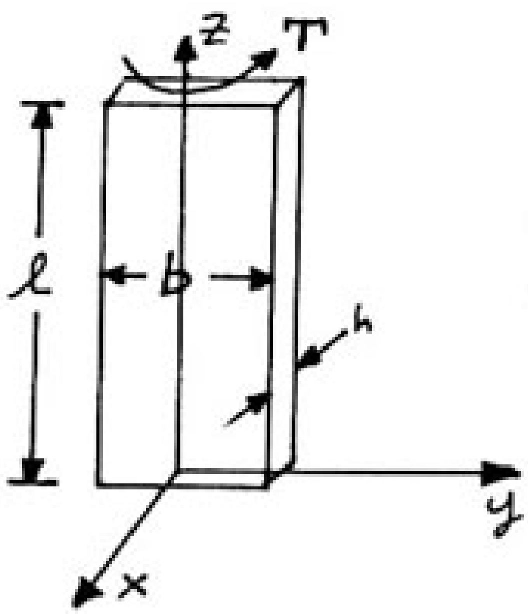
Figure 11.7 **Representation of effect of internal corners**



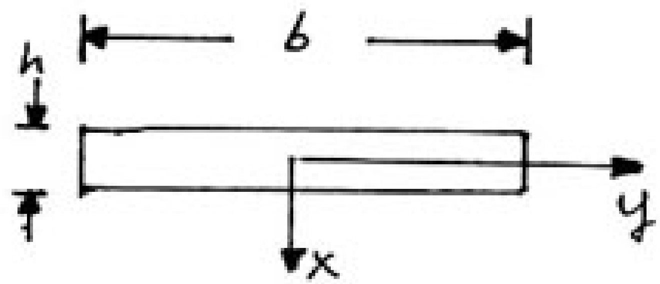
To illustrate some of these points let’s consider specifically…

Torsion of a Narrow Rectangular Cross-Section

Figure 11.8 **Representation of torsion of structure with narrow rectangular cross-section**



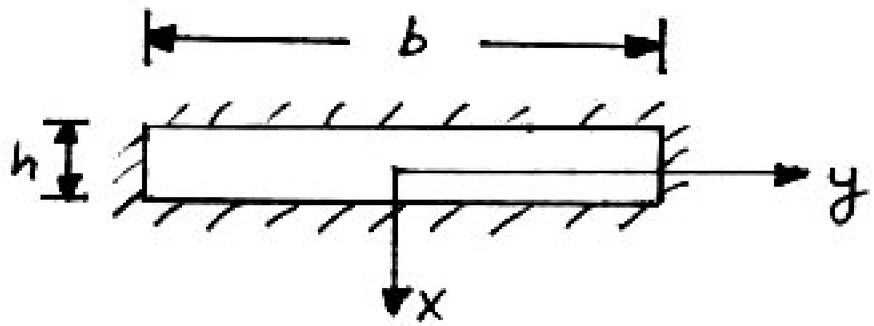
Cross-Section



b >> h

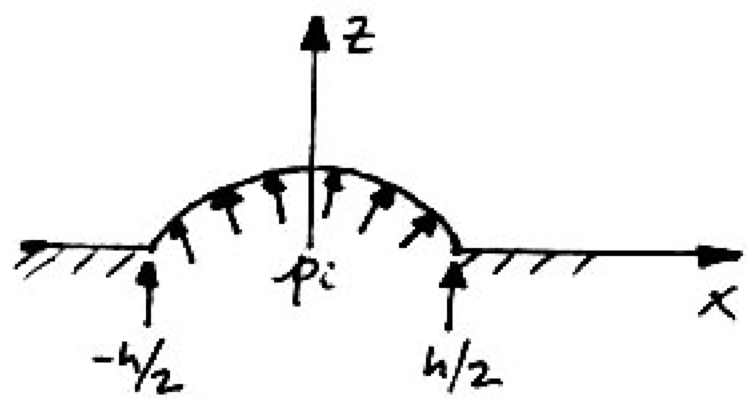
Use the Membrane Analogy for easy visualization:

Figure 11.9 **Representation of cross-section for membrane analogy**



Consider a cross-section in the middle (away from edges):

Figure 11.10 **Side view of membrane under pressure**



The governing Partial Differential Equation. is:

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = - \frac{p_i}{N}$$

Near the middle of the long strip (away from y = ± b/2), we would expect $\frac{\partial^2 w}{\partial y^2}$ to be small. Hence approximate via:

$$\frac{\partial^2 w}{\partial x^2} \approx - \frac{p_i}{N}$$

To get w, let's integrate:

$$\frac{\partial w}{\partial x} \approx - \frac{p_i}{N} x + C_1$$

$$w \approx - \frac{p_i}{2N} x^2 + C_1 x + C_2$$

Now apply the boundary conditions to find the constants:

$$@ \ x = + \frac{h}{2}, \quad w = 0$$

$$\Rightarrow \quad 0 = -\frac{p_i}{2N}\frac{h^2}{4} + C_1\frac{h}{2} + C_2$$

$$@\, x = -\frac{h}{2}, \quad w = 0$$

$$\Rightarrow \quad 0 = -\frac{p_i}{2N}\frac{h^2}{4} - C_1\frac{h}{2} + C_2$$

This gives:

$$C_1 = 0$$

$$C_2 = \frac{p_i h^2}{8N}$$

Thus:

$$\boxed{w \approx \frac{p_i}{2N}\left(\frac{h^2}{4} - x^2\right)}$$

Check the volume:

$$\text{Volume} = \iint w \, dx dy$$

integrating over dy:

$$= b \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{p_i}{2N} \left(\frac{h^2}{4} - x^2 \right) dx$$

$$= \frac{p_i b}{2N} \left[\frac{h^2}{4} x - \frac{x^3}{3} \right]_{-\frac{h}{2}}^{\frac{h}{2}}$$

$$= \frac{p_i b}{2N} \left[\frac{h^2}{4} \frac{2h}{2} - \frac{2}{3} \frac{h^3}{8} \right]$$

$$\Rightarrow \text{Volume} = \frac{p_i b}{N} \frac{h^3}{12}$$

Using the Membrane Analogy:

$$p_i = -k$$

$$N = \frac{1}{2G}$$

$$\text{Volume} = -\frac{T}{2} = \frac{p_i b}{N} \frac{h^3}{12}$$