Lecture 3: Approximate ΔV for Low-Thrust Spiral Climb

Assume initial circular orbit, at $v = v_{co} = \sqrt{\frac{\mu}{r_0}}$.

Thrust is applied tangentially.

Call $a=\frac{F}{M}$.

By conservation of energy, assuming the orbit remains near-circular $\left(v \simeq \sqrt{\frac{\mu}{r}}\right)$,

$$\frac{d}{dt}\left(-\frac{\mu}{2r}\right) \simeq a\sqrt{\frac{\mu}{r}}$$

$$\frac{\mu}{2r^2}\frac{dr}{dt} \simeq a\sqrt{\frac{\mu}{r}}$$

$$\mu^{1/2}\frac{r^{-3/2}}{2}\,dr \simeq a\,dt$$

When we integrate,

$$\int_0^{t_b} a\,dt = \Delta V,$$

and so

$$-\mu^{1/2}r^{-1/2}\Big|_0^{t_b} = \Delta V$$

$$\Delta V = \sqrt{\frac{\mu}{r_0}} - \sqrt{\frac{\mu}{r(t_b)}}$$

or

$$\boxed{\Delta V = v_{co} - v_{c_{final}}} \qquad (1)$$

The result appears to be trivial, but it is not. Notice that the "velocity increment" ΔV is actually equal to the <u>decrease</u> in orbital velocity. The rocket is pushing forward, but the velocity is decreasing. This is because in a r^{-2} force field, the kinetic energy is equal in magnitude but of the opposite sign as the total energy (potential = -$2\times$ kinetic).

If thrust were applied opposite the velocity (negative a), the definition of ΔV would be $\int_0^{t_b}(-a)\,dt$, so the result in general is

$$\boxed{\Delta V = \left|\Delta v_c\right|} \qquad (2)$$

For simplicity, assume now $a = \frac{F}{M}$ = constant, which is actually optimal for many situations. Equation (1) can be recast as

$$\sqrt{\frac{\mu}{r_0}} - \sqrt{\frac{\mu}{r}} = at$$

and solving for r,

$$\left[r = \frac{r_0}{\left(1 - \frac{at}{\sqrt{\mu/r_0}}\right)^2} = \frac{r_0}{\left(1 - \frac{at}{v_{co}}\right)^2} \right] \qquad (3)$$

This shows how the radial distance "spirals out" in time. In principle, this says $r \rightarrow \infty$ at $t = \frac{v_{co}}{a}$, a crude indication of "escape". But of course, the orbit is no longer "near-circular" when approaching escape, so this result is not precise. One can get some improvement for the estimation of escape ΔV as follows.

The radial velocity $\dot{r}$ can be calculated from (3) by differentiation. Notice that this is in the nature of an iteration, since $\dot{r}$ was implicitly ignored in the energy balance which led to (3). We obtain

$$\dot{r} = \frac{2a/v_{co}\ r_0}{\left(1 - at/v_{co}\right)^3} \qquad (4)$$

The tangential component $v_\theta = r\dot{\theta}$ is still approximated as the orbital velocity, i.e.,

$$r\dot{\theta} = \sqrt{\frac{\mu}{r}} = \sqrt{\frac{\mu}{r_0}} - at = v_{co}\left(1 - \frac{at}{v_{co}}\right) \qquad (5)$$

The overall kinetic energy is therefore

$$\frac{v^2}{2} = \dot{r}^2 + \left(r\dot{\theta}\right)^2 = \frac{v_{co}^2}{2}\left[\left(1 - \frac{at}{v_{co}}\right)^2 + \frac{4\left(\frac{ar_0}{v_{co}^2}\right)^2}{\left(1 - \frac{at}{v_{co}}\right)^6}\right] \qquad (6)$$

The escape point is defined by its having zero total energy, i.e.,

$$\frac{v_e^2}{2} = \frac{\mu}{r_e},$$

or

$$\frac{1}{2}\left(\frac{v_e}{v_{co}}\right)^2 - \frac{r_0}{r_e} = 0.$$

Substituting,

$$\frac{1}{2}\left(1-\frac{at}{v_{co}}\right)^2 + \frac{2\left(\frac{ar_0}{v_{co}^2}\right)^2}{\left(1-\frac{at}{v_{co}}\right)^6} - \left(1-\frac{at}{v_{co}}\right)^2 = 0$$

or

$$\frac{4\left(ar_0/v_{co}^2\right)^2}{\left(1-\frac{at}{v_{co}}\right)^6} = \left(1-\frac{at}{v_{co}}\right)^2$$

$$1-\frac{at}{v_{co}} = \left(\frac{2ar_0}{v_{co}^2}\right)^{1/4} = \left(\frac{2a}{\mu/r_0^2}\right)^{1/4} = (2\nu)^{1/4} \qquad (7)$$

where

$$\nu = \frac{a}{\mu/r_0^2} \qquad (8)$$

is the ratio of thrust to gravity, a small number for us.

Since $at=\Delta V$, Equation (7) reads

$$\boxed{\Delta V_{esc.} \cong v_{co}\left[1-(2\nu)^{1/4}\right]} \qquad (9)$$

In a more rigorous analysis, the factor $2^{1/4} = 1.19$ turns out to be actually 0.79:

$$\boxed{\Delta V_{esc.} = v_{co}\left[1-0.79\nu^{1/4}\right]} \qquad (10)$$

(more exact)

LOW THRUST TO ESCAPE: RESULTS OF NUMERICAL COMPUTATIONS

DEFINITIONS:

$$\nu = \frac{\left(F/m\right)}{\left(\mu/r_0^2\right)}$$

$$v_{co} = \sqrt{\frac{\mu}{r_0}}$$

$$s_{esc.} = s\left(E=0\right) = \frac{\mu}{2r_0\left(F/M\right)} = \frac{v_{co}^2}{2a} \quad \text{(linear distance travelled)}$$

$$\left(a = \frac{F}{M}\right)$$

ν	$\frac{T_{esc}}{r_0}$	$\left(\frac{dr}{ds}\right)_{esc.}$	$\frac{\Delta V}{v_{co}}$	$\frac{s_{esc.}}{r_0}$	$\frac{r_{esc.}}{r_0}\sqrt{\nu}$	$\frac{1-\frac{\Delta V}{v_{co}}}{\nu^{1/4}}$
10^{-2}	8.9	0.63	0.75	50	0.89	0.79
10^{-3}	28	0.63	0.86	500	0.885	0.79
10^{-4}	88	0.63	0.92	5,000	0.88	0.80
10^{-5}	278	0.64	0.96	50,000	0.88	0.71

So, to a good approximation,

$$\boxed{\frac{\left(\Delta V\right)_{esc}}{v_{co}} \simeq 1 - 0.79\,\nu^{1/4}}$$

$$\boxed{\frac{r_{esc}}{r_0} \simeq \frac{0.88}{\sqrt{\nu}}}$$

$$\boxed{\left(\frac{dr}{ds}\right)_{esc} \simeq 0.63}$$

Edelbaum's Sub-Optimal Climb and Plane Change

Instead of optimizing the tilt profile $\alpha(\theta)$, Edelbaum (1961, 1973) just kept $|\alpha|$ constant during each orbit, then optimized $|\alpha|(R)$.

So,

$$\alpha(\theta) = \begin{cases} +\alpha & \text{for } -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\ -\alpha & \text{for } \frac{\pi}{2} < \theta < \frac{3\pi}{2} \end{cases} \tag{31}$$

We still have (Equation 1)

$$\frac{di}{d\theta} = \frac{F}{M}\frac{R^2}{\mu}\sin\alpha\cos\theta,$$

but now (using $\alpha \equiv 1$)

$$\left\langle \sin\alpha\cos\theta \right\rangle_\theta = \frac{1}{\pi}\sin\alpha\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\theta\, d\theta = \frac{2}{\pi}\sin\alpha$$

$$\therefore \boxed{\left\langle \frac{di}{d\theta} \right\rangle = \frac{2}{\pi}\frac{R^2}{\mu}\frac{F}{M}\sin\alpha} \tag{32}$$

Similarly, we still have

$$\boxed{\frac{dR}{d\theta} = \frac{2FR^3}{M\mu}\cos\alpha} \tag{33}$$

which needs no averaging.

Dividing (33) by (32) and dropping the averaging sign,

$$\boxed{\frac{di}{dR} = \frac{\tan\alpha}{\pi R}} \tag{34}$$

We also have

$$\frac{d\Delta V}{dR} = \frac{F/M}{\left(\frac{dR}{dt}\right)},$$

and now

$$\frac{dR}{dt} = 2\frac{F}{M}\frac{R^{3/2}}{\mu^{1/2}}\cos\alpha,$$

$$\boxed{\frac{d\Delta V}{dR} = \frac{1}{2}\frac{\sqrt{\mu/R^3}}{\cos\alpha}} \tag{35}$$

To optimize $\alpha(R)$, we again look for $\min.\frac{d\Delta V}{dR}$ for a given $\frac{di}{dR}$.

For the Hamiltonian

$$H = \frac{1}{2}\frac{\sqrt{\mu/R^3}}{\cos\alpha(R)} + \lambda_i\left(\frac{di}{dR} - \frac{\tan\alpha(R)}{\pi R}\right) \quad (36)$$

The "control" variable is α, the "stable variable" is i and the independent variable (replacing time) is R.

The optimality conditions are

$$\left.\begin{aligned} \frac{\partial H}{\partial \alpha} &= 0 \\ \frac{d\lambda_i}{dR} &= \frac{\partial H}{\partial i} \end{aligned}\right\} \quad (37)$$

and the "transversality condition"

$$\left[\lambda_i^{\delta_i}\right]_{R_1}^{R_2} = 0$$

is satisfied automatically, because i is prescribed at both ends.

From (37b),

$$\frac{d\lambda_i}{dR} = 0 \qquad \boxed{\lambda_i = \text{constant}} \quad (38)$$

and from (37a), using $\frac{1}{\cos\alpha} = \sqrt{1+\tan^2\alpha}$,

$$\frac{1}{2}\sqrt{\frac{\mu}{R^3}}\frac{\tan\alpha}{\sqrt{1+\tan^2\alpha}} - \frac{\lambda_i}{\pi R} = 0 \qquad \Rightarrow \qquad \sin\alpha = \frac{2\lambda_i}{\pi}\sqrt{\frac{R}{\mu}} \quad (39)$$

Pending determination of λ_i, (39) indicates that the thrust tilt amplitude α increases over the mission, so that most of the plane-change activity is deferred to the last part of the climb, when the orbital velocity is lower.

To find λ_i, use the constraint on the total Δ_i.

From (34),

$$\Delta_i = \int_{R_1}^{R_2}\frac{\tan\alpha}{\pi R}dR \quad (40)$$

From (39),

$$R = \mu\left(\frac{\pi}{2\lambda_i}\right)^2 \sin^2\alpha,$$

so

$$\frac{dR}{R} = \frac{2d(\sin\alpha)}{\sin\alpha} = \frac{2d\alpha}{\tan\alpha}.$$

Hence,

$$\Delta_i = \int_{\alpha_1}^{\alpha_2} \frac{2}{\pi} d\alpha$$

$$\boxed{\Delta_i = \frac{2}{\pi}\left(\alpha_2 - \alpha_1\right)} \qquad (41)$$

A separate relationship between α_1 and α_2 comes from (39):

$$\boxed{\frac{\sin\alpha_2}{\sin\alpha_1} = \sqrt{\frac{R_2}{R_1}} = \frac{v_1}{v_2}} \qquad (42)$$

where v_1, v_2 denote the initial and final orbital velocities.

Combining (41) and (42),

$$\frac{\sin\left(\alpha_1 + \frac{\pi}{2}\Delta i\right)}{\sin\alpha_1} = \frac{v_1}{v_2};$$

$$\cos\left(\frac{\pi}{2}\Delta i\right) + \cot\alpha_1 \sin\left(\frac{\pi}{2}\Delta i\right) = \frac{v_1}{v_2}$$

$$\Rightarrow \boxed{\cot\alpha_1 = \frac{\frac{v_1}{v_2} - \cos\frac{\pi}{2}\Delta i}{\sin\frac{\pi}{2}\Delta i}} \qquad \left(\text{or } \sin\alpha_1 = \frac{\sin\frac{\pi}{2}\Delta i}{\sqrt{1 + \left(\frac{v_1}{v_2}\right)^2 - 2\frac{v_1}{v_2}\cos\frac{\pi}{2}\Delta i}}\right) \qquad (43)$$

The most important quantity is the optimized ΔV.

From (35),

$$\Delta V = \frac{1}{2}\int_{R_1}^{R_2} \sqrt{\frac{\mu}{R^3}} \frac{dR}{\cos\alpha} = \frac{1}{2}\int_{R_1}^{R_2} \sqrt{\frac{\mu}{R}} \frac{dR/R}{\cos\alpha}$$

$$= \frac{1}{2}\int_{R_1}^{R_2} \left(\frac{2\lambda_i}{\pi\sin\alpha}\right)\left(\frac{1}{\cos\alpha}\right)\left(\frac{2d\alpha}{\tan\alpha}\right) = \frac{2\lambda_i}{\pi}\int_{\alpha_1}^{\alpha_2} \frac{d\alpha}{\sin^2\alpha} = \frac{2\lambda_i}{\pi}\left(\cot\alpha_1 - \cot\alpha_2\right) \qquad (44)$$

Putting (from 39)

$$\frac{2\lambda_i}{\pi} = v_1 \sin\alpha_1,$$

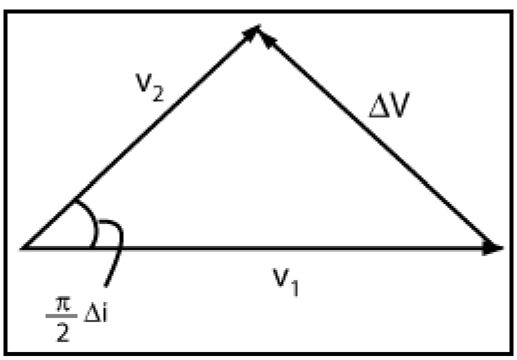
and using, analog to (43),

$$\cot\alpha_2 = \frac{\cos\frac{\pi}{2}i - \frac{v_2}{v_1}}{\sin\frac{\pi}{2}\Delta i},$$

$$\Delta V = v_1 \frac{\sin\frac{\pi}{2}\Delta i}{\sqrt{1+\left(\frac{v_1}{v_2}\right)^2 - 2\frac{v_1}{v_2}\cos\frac{\pi}{2}\Delta i}} \; \frac{\left(\frac{v_1}{v_2} - \cos\frac{\pi}{2}\Delta i\right) - \left(\cos\frac{\pi}{2}\Delta i - \frac{v_2}{v_1}\right)}{\sin\frac{\pi}{2}\Delta i}$$

and simplifying,

$$\boxed{\Delta V = \sqrt{v_1^2 + v_2^2 - 2v_1v_2\cos\left(\frac{\pi}{2}\Delta i\right)}} \qquad (45)$$



Geometrically, ΔV appears as the <u>vector</u> difference of $\bar{v}_1$ and $\bar{v}_2$, except the angle between $\bar{v}_1$ and $\bar{v}_2$ is not the actual Δi, but $\frac{\pi}{2}\Delta i$. The extra factor reflects the inefficiency $\left(\frac{1}{\eta_{NSSK}}\right)$ associated with thrusting through the full 180° in each out-of-plane direction.

<u>Example</u>

Consider again LEO (400 Km) to GEO, with

$\Delta i = 28.5^\circ$;
$v_1 = 7673$ m/s,
$v_2 = 3072$ m/s.

From (46),

$$\Delta V = \sqrt{(7673)^2 + (3072)^2 - 2\times 7673\times 3072\cos\left(\frac{\pi}{2}28.5^\circ\right)}$$

$\boxed{\Delta V = 5903 \text{ m/s}}$

This is noticeably worse than the true optimum $\Delta V = 5768 \text{ m/s}$ calculated for the case when $\alpha(\theta)$ is also modulated as $\tan\alpha \sim \cos\theta$.

The initial and final tilt angles are

$$\sin\alpha_1 = \frac{\sin\left(\frac{\pi}{2}28.5^\circ\right)\times 3072}{5903} = 0.3665 \longrightarrow \boxed{\begin{array}{l}\alpha_1 = 21.5^\circ \\ \alpha_2 = 66.3^\circ\end{array}}$$

$$\alpha_2 = \alpha_1 + \frac{\pi}{2}\Delta i$$

These are smaller than the peak values $\alpha_{1_{MAX}} = 30.5^\circ, \alpha_{2_{MAX}} = 72.2^\circ$ in this optimal case, but, of course, they are applied for the whole half-orbit.