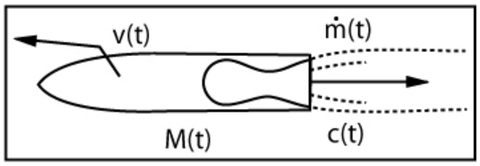
Lecture 1b: Review of Rocket Propulsion

The only practical way to accelerate something in free space is by reaction. The idea is the same as in air breathing propulsion (to push something backwards) but in rockets the "something" must be inside and is lost.

Here is a revealing derivation of the thrust equation for <u>vacuum</u>:



$$\text{Mom.}(t) = M(t)\,v(t) + \int_0^t \dot m(t')\left[v(t') - c(t')\right] dt' = \text{Constant}$$

no drag, no interaction of molecules with ambient air

$$\frac{d(\text{Mom})}{dt} = 0$$

$$M\frac{dv}{dt} + v\frac{dM}{dt} + \dot m(v - c) = 0$$

But

$$\dot m = -\frac{dM}{dt} \rightarrow \boxed{M\frac{dv}{dt} = \dot m c}$$

call this thrust, F.

Notice

$$\frac{d(Mv)}{dt} = F - v\dot m = \dot m(c - v)$$

which can be (+) or (-)

Using the same technique, the kinetic energy of the system rocket-jet is

$$KE = \frac{1}{2}Mv^2 + \int_0^t \frac{1}{2}\dot m(t')\left[v(t') - c(t')\right]^2 dt'$$

So,

$$\frac{dKE}{dt} = \underbrace{\cancel{Mv\frac{dv}{dt}}}_{\dot m c v} + \cancel{\frac{1}{2}v^2}\underbrace{\frac{dM}{dt}}_{-\dot m} + \frac{1}{2}\dot m\left(\cancel{v^2} + c^2 - \cancel{2vc}\right) = \frac{1}{2}\dot m c^2$$

So, if thermal (or electrical) energy is expended internally at the rate $\dot E$, and converted to total kinetic energy with efficiency η_{th} (or $\eta_{el.}$) ;
i.e.,

$$\frac{d(KE)}{dt} = \eta_{th}\dot{E},$$

then

$$\boxed{\eta_{th}\dot{E} = \frac{1}{2}\dot{m}c^2}$$

<u>jet kinetic power</u>

Note that we counted both vehicle <u>and</u> wake KE as produced by $\dot{E}$, and this is unambiguous. If we try to define "useful Propulsive work" as $Fv = \dot{m}cv$, then we find that the "propulsive efficiency"

$$\eta_{prop} = \frac{F.v}{\eta_{th}\dot{E}} = \frac{\dot{m}vc}{\frac{1}{2}\dot{m}c^2} = \frac{2v}{c}$$

is arbitrarily high! $\left(\text{If } v > \frac{c}{2}, \eta_{pr.} > 1\right)$.

For this reason, η_{prop} is not used for rockets. But it is still true that thrusting at high speed increases kinetic energy more

$$\Delta\left(\frac{1}{2}m_f v^2\right) = \frac{1}{2}m_f\left[\left(v_0 + c\ln\frac{m_0}{m_f}\right)^2 - v_0^2\right]$$

$$= m_f\left[\frac{1}{2}c^2\ln^2\frac{m_0}{m_f} + v_0 c\ln^2\frac{m_0}{m_f}\right]$$

In the presence of external air, some modification is needed, leading to the well-known formula (or finite P_e)

$$F = \dot{m}u_e + A_e(P_e - P_a) \equiv \dot{m}c$$

defines $c \simeq u_e$ = Jet speed far from exhaust

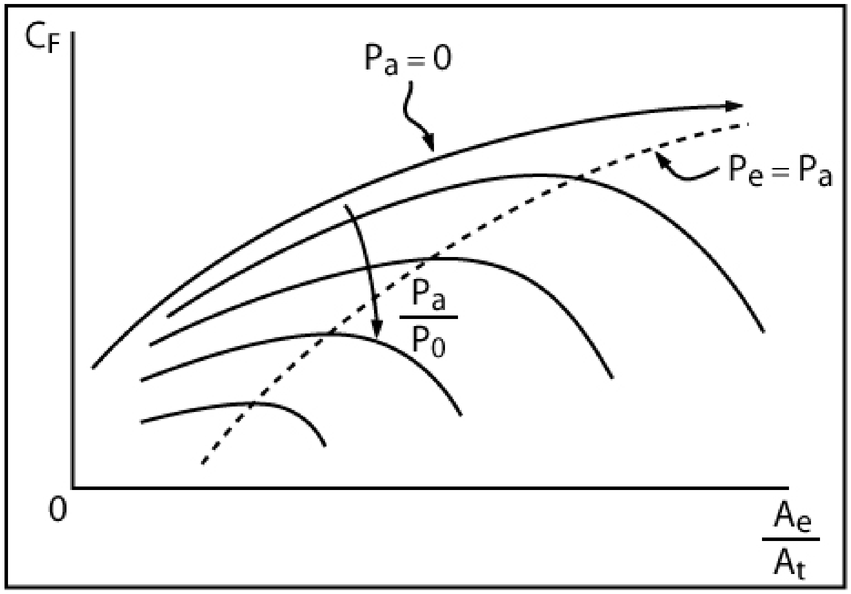
For finite P_a, in thermal rockets, increasing A_e increases u_e (towards a limit $u_{e_{max}} \cong \sqrt{2c_pT_0}$), but it eventually makes $(P_e - P_a)A_e$ <u>negative</u>.

The best A_e is such as to make $P_e = P_a$.

The thrust coefficient c_F is used to quantify the performance of nozzles. Starting from

$$F = \dot{m}u_e + A_e(P_e - P_a)$$

Isentropic flow $p \propto p^{r}$ $p^{V^{r}}$ = constant



and assuming ideal gas,

$$\dot{m} = \rho_t u_t A_t = \rho_0 \left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}} \sqrt{\gamma R T_0 \left(\frac{2}{\gamma-1}\right)}\, A_t$$

$$\frac{P_0}{R_0 T_0}$$

$$\Rightarrow \dot{m} = \frac{P_0 A_t}{c^*}$$

with

$$c^* = \frac{\sqrt{RT_0}}{\Gamma(\gamma)}, \quad \Gamma = \sqrt{\gamma}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{2(\gamma-1)}}$$

$$\rightarrow F = P_0 A_t \left[\underbrace{\frac{u_e}{c^*} + \frac{A_e}{A_t}\left(\frac{P_e}{P_0} - \frac{P_a}{P_0}\right)}_{\text{non-dimensional } c_F}\right]$$

So

$$F = c_F P_0 A_t$$

Usually, $c_F \sim 1.5 - 2$

Specific impulse (inverse of specific fuel consumption) is defined as

$$I_{sp} = \frac{F}{\dot{m}g} = \frac{c}{g} = \frac{c^*c_F}{g}$$

We want high I_{sp} (high c), since

$$m\frac{dv}{dt} = -\frac{dm}{dt}c \rightarrow \Delta V = c\ln\frac{m_0}{m_f}$$

Using

$$m_0 = m_P + m_S + m_L, \qquad (m_P = \text{propellant},\ m_S = \text{structure},\ m_L = \text{payload})$$

$$m_L = \frac{m_P}{e^{\frac{\Delta V}{c}} - 1} - m_S \quad \text{or} \quad m_L = m_0 e^{-\frac{\Delta V}{c}} - m_S$$

So, for given m_P, m_S, the higher c (I_{sp}), the higher m_L; this dependence is <u>at least</u> linear (for small $\frac{\Delta V}{c}, m_L \simeq m_P\frac{c}{\Delta V} - m_S$), but it becomes exponentially fast for high energy missions ($\frac{\Delta V}{c} \gg 1$).

For chemical rockets, c is limited by chemical energy/mass in fuel $c_{max} = \sqrt{2E}$.

In general, since

$$\frac{1}{2}\dot{m}u_e^2 = \eta_{th}\dot{E} = \eta_{th}\dot{m}E$$

$$u_e = \sqrt{2\eta_{th}E}$$

(Accounting for work of expansion leads to a replacement of E by H, the enthalpy.)

For ideal expansion, Brayton cycle, so

$$\eta_{th} = 1 - \left(\frac{P_e}{P_0}\right)^{\frac{\gamma-1}{\gamma}}$$

and

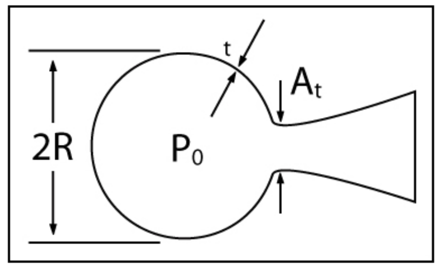
$H=c_PT_0$ (ideal gas)

$$u_e = \sqrt{2c_PT_0\left[1 - \left(\frac{P_e}{P_0}\right)^{\frac{\gamma-1}{\gamma}}\right]}$$

This shows we want high P_0/P_e. This means large area ratio $\frac{A_e}{A^*}$, not necessarily high P_0. In fact, for vacuum operation, higher P_0 simply means higher P_e, with no increase of u_e (or I_{sp}). In a closer look, higher P_0 can increase T_0 by inhibiting dissociation $\longrightarrow$ higher I_{sp}.

But the main reason to go to high P_0 is to <u>reduce weight</u> for a given thrust (also for boosting, where P_e cannot be much lower than P_a, so high P_0/P_e means high P_0).

Roughly



$$\frac{M}{F} \simeq \frac{4\pi R^2 t\,\rho_w}{c_F P_0 A_t} \qquad 2\pi R t\sigma = \pi R^2 p_0$$

$$t = p_0 \frac{R}{2\sigma}$$

$$\frac{M}{F} \simeq \frac{4\pi \cancel{R^2} P_0 R \rho_w}{2\sigma c_F P_0 K \cancel{R^2}} \qquad A_t = K R^2$$

$$\frac{M}{F} \simeq \frac{2\pi}{K c_F} \frac{\rho_w}{\sigma} R \qquad F = c_F P_0 K R^2 \qquad R = \sqrt{\frac{F}{c_F P_0 K}}$$

$$\frac{M}{F^{3/2}} = \frac{2\pi}{(K c_F)^{3/2}} \frac{\rho_w}{\sigma} \sqrt{\frac{1}{P_0}}$$

Thus, for a given thrust level, the engine mass scales like $1/\sqrt{P_0}$.

For a given <u>total impulse</u> (not thrust), it may be better to reduce P_0, reduce thrust, operate longer. In boosters, there is a complex tradeoff, involving gravity losses, drag penalties, improved I_{sp} at high P_0, etc. In general, for boosters it is found advantageous to go to high P_0, limited mostly (in liquid rockets) by turbomachinery. For space engines the result is less clear, but they do tend to optimize at much lower P_0.

The power of a rocket can be extremely high. For the Shuttle

$F \simeq 3000 \text{ ton} = 3\times 10^7 \text{Nt}$,

$\bar{c} \simeq 3300 \text{ m/sec}$

$$\frac{1}{2}\dot{m}c^2 = \frac{1}{2}Fc = \frac{1}{2}3\times 10^7 \times 3.3\times 10^3 = 5\times 10^{10} \text{ watt} = 50 \text{ GW!!}$$

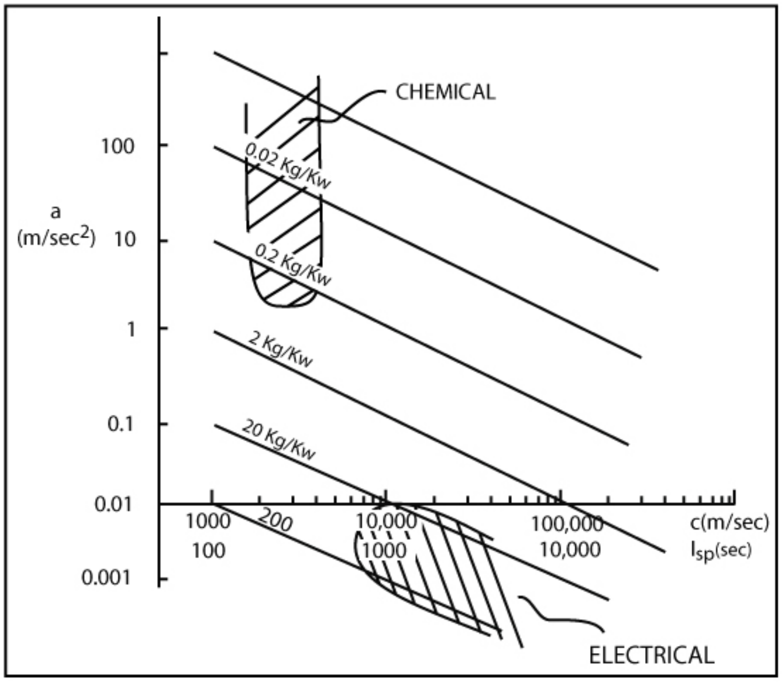
Thus, if one step in the power chain involves electrical power, the engine is likely to be very heavy. Why then electrical? Because it breaks the u_e limit, allowing any I_{sp}, or, in other words, it gives very fuel efficient rockets. With EM forces one can increase u_e almost arbitrarily; however looking at the power requirements,

$$P = \frac{1}{\eta_c}\frac{1}{2}\dot{m}c = \frac{Fc}{2\eta_c}$$

$$\frac{P}{M} = \frac{F}{M}\frac{c}{2\eta_c} = \frac{ac}{2\eta_c}$$

$0.5 \text{ watt/Kg} \rightarrow 2 \text{ Kg/watt} = 2000 \text{ Kg/KW}$

$50 \text{ watt/Kg} \rightarrow 2 \times 10^{-2} \quad \rightarrow 20$



So, with reasonable mass/power ratios for electric power one gets very low accelerations

If one pushes I_{sp} in an electric thruster, M_P is reduced for given ΔV, but M_s increases due to the high P, unless a is very low (which may be impossible because of mission duration constraints). Thus, an optimum I_{sp} is found to exist (~2000 sec, depending on conditions).

Reference

Martinez-Sanchez, M., and J. E. Pollard. "Spacecraft Electric Propulsion – An Overview." *Journal of Propulsion and Power* 14, no. 5 (September–October 1998): 688-699. [A publication of the American Institute of Aeronautics and Astronautics, Inc.]