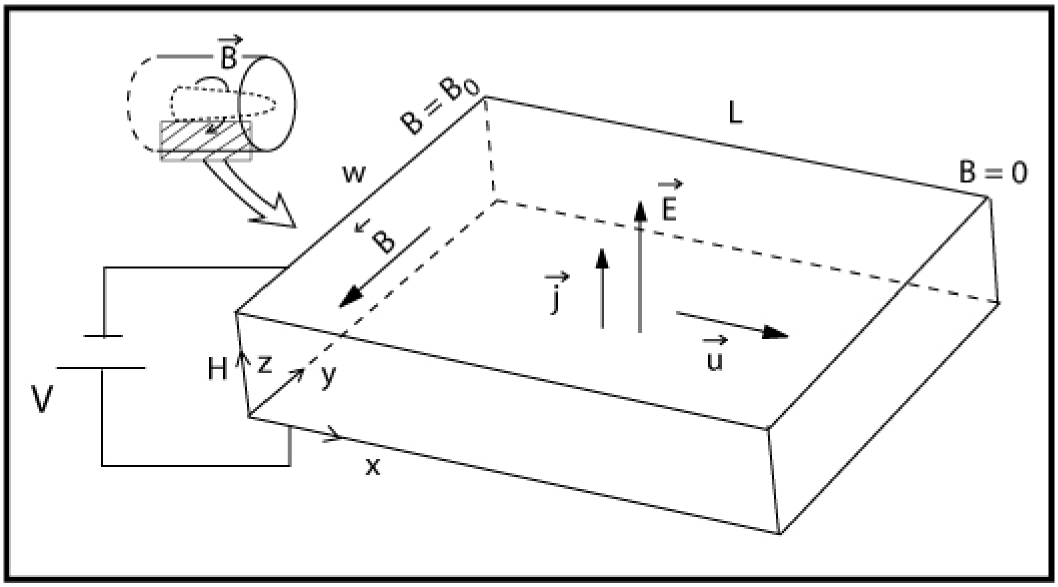
Lecture 22: A Simple Model For MPD Performance-onset

It is well known that rapidly pulsed current tends to concentrate near the surface of copper conductors forming a "skin". A similar effect occurs when current flows through a highly conductive and rapidly moving plasma: current tends to concentrate near the entrance and exit of the channel. The reason is the appearance of a strong back EMF which tends to block current over most of the channel's length. This is most easily seen if we "unwrap" the annular chamber of an MPD thruster into a rectangular 1-D channel.



Ampère's law:

$$\vec{j} = \frac{1}{\mu_0}\nabla \times \vec{B} \qquad (1)$$

In our case $\nabla = \frac{\partial}{\partial x}\vec{I}_x$,

so $j_z \equiv j = +\frac{1}{\mu_0}\frac{dB_y}{dx}$

and calling

$$B \equiv -B_y, \qquad j = -\frac{1}{\mu_0}\frac{dB}{dx} \qquad (2)$$

Ohm's law (ignoring Hall effect) is

$$\vec{j} = \sigma\left(\vec{E} + \vec{u} \times \vec{B}\right) \qquad (3)$$

or, using

$$E \equiv E_z = \frac{V}{H}, \quad B \equiv -B_y, \quad u = u_x$$

$$j = \sigma\left(E - uB\right) \tag{4}$$

Combining (2) and (4),

$$\frac{dB}{dx} = -\sigma\mu_0\left(E - uB\right) \tag{5}$$

The flow velocity u evolves along x according to the momentum equation (ignoring pressure forces)

$$\dot{m}\frac{du}{dx} + A\underbrace{\cancel{\frac{dP}{dx}}}_{\text{neglect for now}} = \left(\vec{j}\times\vec{B}\right)_x A = jBwH \tag{6}$$

Substitute (2) into (6):

$$\dot{m}\frac{du}{dx} = -\frac{1}{\mu_0}\frac{dB}{dx}B\,wH = -\frac{wH}{\mu_0}\frac{d}{dx}\left(\frac{B^2}{2}\right) \tag{7}$$

Integrate:

$$\dot{m}u + wH\frac{B^2}{2\mu_0} = \underbrace{\cancel{\dot{m}u_0}}_{\text{neglect}} + wH\frac{B_0^2}{2\mu_0}$$

$$u = \frac{wH}{2\mu_0}\frac{B_0^2 - B^2}{\dot{m}} \tag{8}$$

Putting this in Equation (5),

$$\frac{dB}{dx} = -\sigma\mu_0\left[E - \frac{wH}{2\mu_0\,\dot{m}}B\left(B_0^2 - B^2\right)\right] \tag{9}$$

If we approximate the conductivity σ as a constant, this can be integrated as

$$\sigma\mu_0 x = \int_B^{B_0}\frac{dB}{E - \frac{wH}{2\mu_0\,\dot{m}}B(B_0^2 - B^2)} \tag{10}$$

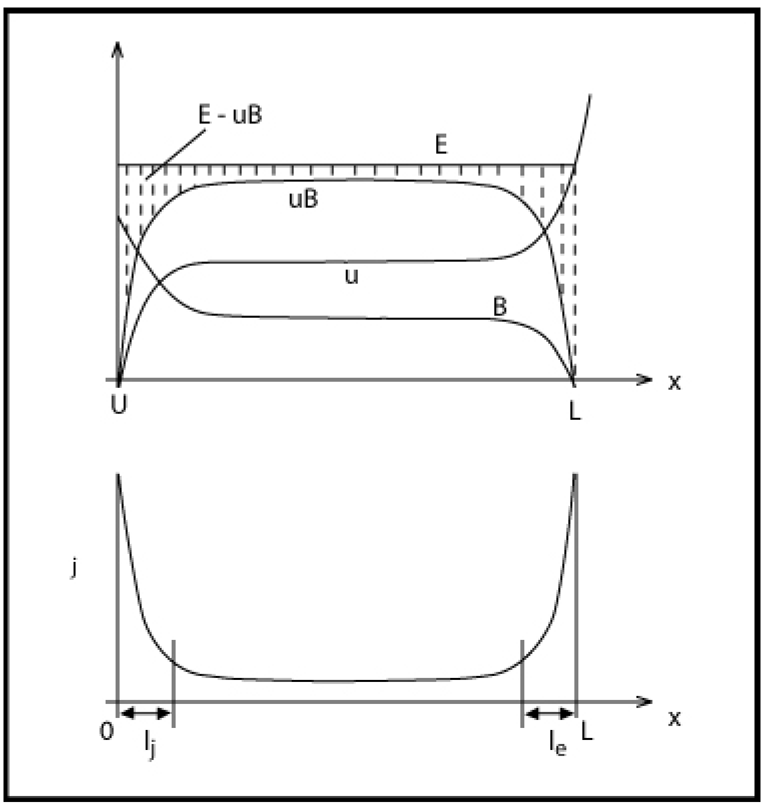
This integral can actually be calculated analytically, but the resulting expression is not very transparent. It is more useful to examine its behavior qualitatively. The denominator in the integrand is the driving field (applied field E, minus back emf, uB). The field B_0 at x=0 is a measure of the current I, because integrating (2) between x=0 and x=1 gives

$$\int_0^L j\,dx = \frac{I}{w} = \frac{B_0}{\mu_0} \Rightarrow B_0 = \frac{\mu_0 I}{w} \tag{11}$$

On the other hand, carrying (10) all the way to x=L, gives

$$\sigma\mu_0 L = \int_0^{B_0} \frac{dB}{E - \frac{wH}{2\mu_0 \dot{m}} B(B_0^2 - B^2)} \tag{12}$$

where, once I and $\dot{m}$ are specified, only E remains as an unknown. This is then the equation for voltage V=EH. Consider conditions where the maximum value of the back emf uB reaches almost the level E. This means that the integrand will be very large as long as this condition prevails, and it must indicate a large value of $\sigma\mu_0 L$. By the same token, Equation (5) says that B will remain flat when E-uB<<E, and from (2), there will be little current in this region. Schematically:



We see here that two strong current concentrations develop, near x=0 and x=L. Let us investigate <u>when</u> this situation will arise. From (12), the denominator is minimum at a B value that maximizes $B(B_0^2 - B^2)$, namely, $B_0^2 - 3B^2 = 0$

$$B_1 \equiv B_{(uB)_{MAX}} = \frac{B_0}{\sqrt{3}} \tag{13}$$

and then

$$\left(E - uB\right)_{MIN} = E - \frac{wH}{2\mu_0 \dot{m}} \frac{2B_0^3}{3\sqrt{3}},$$

and, since by assumption, this difference is much less than E, we find

$$E \simeq \frac{wH}{\cancel{2}\mu_0 \dot{m}} \frac{\cancel{2}B_0^3}{3\sqrt{3}} \tag{14}$$

or, in terms of voltage and current,

$$V \simeq \frac{1}{3\sqrt{3}}\left(\frac{H}{w}\right)\mu_0^2 \frac{I^3}{\dot{m}} \tag{15}$$

Returning now to Equation (5), we notice that near both x=0 and x=L, uB<<E, so $\frac{dB}{dx} \simeq -\sigma\mu_0 E$, and so the thickness l of the thin current layers (where B varies substantially) can be estimated as follows:

$$l_0 \simeq \frac{B_0 - B_1}{\sigma\mu_0 E} \quad ; \quad l_e \simeq \frac{B_1}{\sigma\mu_0 E} \tag{16}$$

where $B_1 = \frac{B_0}{\sqrt{3}}$.

Using (16),

$$l_0 \simeq \frac{3\left(\sqrt{3} - 1\right)\dot{m}}{\sigma w H B_0^2} \;;\; l_e \simeq \frac{3\dot{m}}{\sigma w H B_0^2} \tag{17}$$

Remembering (from (8)) that the exit velocity is

$$u_e = \frac{wH}{2\mu_0}\frac{B_0^2}{\dot{m}} = \frac{1}{2}\frac{H}{w}\frac{\mu_0 I^2}{\dot{m}} \tag{18}$$

these results can be written as

$$\sigma\mu_0 u_e l_0 \simeq \frac{3}{2}\left(\sqrt{3} - 1\right) \quad ; \quad \sigma\mu_0 u_e l_0 \simeq \frac{3}{2} \tag{19}$$

The non-dimensional group $\sigma\mu_0 ul$ is called the Magnetic Reynolds Number (R_m) (based on length l). What we have seen is that this R_m, when based on the current layer thickness, is of order unity. Since we started out by assuming conditions when these layers are thin, i.e., l_e, l_0<<L, we can now state that this will occur when

$$R_m(L) \equiv \sigma\mu_0 u_e L >> 1 \tag{20}$$

This is indeed the condition for operation in the pure MPD regime.

Effects of Dissipation

The high-current inlet and exit layers are very dissipative. Their resistances can be estimated as

$$R_0 = \frac{\frac{4}{3}H}{\sigma w l_0} \quad ; \quad R_e = \frac{\frac{4}{3}H}{\sigma w l_e} \tag{21}$$

(the 4/3 factor accounts for the "triangular" current distribution in the layers) and so the Ohmically dissipated power is

$$D = I_0^2 R_0 + I_e^2 R_e \tag{22}$$

where

$$I_0 = \frac{w}{\mu_0}(B_0 - B_1) = \frac{wB_0}{\mu_0}\left(1 - \frac{1}{\sqrt{3}}\right) = I\left(1 - \frac{1}{\sqrt{3}}\right) \tag{23}$$

and

$$I_e = I - I_0 = I/\sqrt{3} \tag{24}$$

Substituting, we find

$$D_0 = I^2\left(1 - \frac{1}{\sqrt{3}}\right)^2 \frac{4}{3}H \frac{\sigma H \mu_0^2 I^2}{\sigma w 3\left(\sqrt{3} - 1\right)\dot{m} w} = \frac{4}{3}\frac{1 - \frac{1}{\sqrt{3}}}{3\sqrt{3}}\left(\frac{H}{w}\right)^2 \frac{\mu_0^2 I^4}{\dot{m}}$$

$$D_e = I^2 \frac{1}{3} \frac{\frac{4}{3}H}{\sigma w} \frac{\sigma H \mu_0^2 I^2}{3w\dot{m}} = \frac{4}{3}\frac{1}{9}\left(\frac{H}{w}\right)^2 \frac{\mu_0^2 I^4}{\dot{m}}$$

and, in total,

$$D = \frac{4}{9\sqrt{3}}\left(\frac{H}{w}\right)^2 \frac{\mu_0^2 I^4}{\dot{m}} \tag{25}$$

Part of this dissipation goes to heating the gas, but the major portion is used in ionizing and exciting (followed by radiation) the gaseous atoms. Let $eV_i^{'} \simeq 2$ to 3 times (eV_i) be the effective ionization energy per atom, and α_e the degree of ionization at the exit. We then have

$$D \simeq \alpha_e eV_i' \frac{\dot{m}}{m_i} \tag{26}$$

or, from (25),

$$\alpha_e \simeq \frac{m_i}{eV_i'} \frac{4}{9\sqrt{3}} \left(\frac{H}{w} \frac{\mu_0 I^2}{\dot{m}} \right)^2 \tag{27}$$

This indicates very rapid increase of the ionization fraction as the current increases, or as the flow is reduced.

<u>Instability "onset"</u>

For a given thruster, as $I^2/\dot{m}$ is increased, α_e increases rapidly. When it reaches unity, the behavior of the plasma near the exit changes drastically. This is because any extra dissipation cannot be absorbed into ionization anymore, and goes instead directly into heating the plasma (or perhaps the electron component only). This causes conductivity to increase whenever the current concentrates, which leads to further current concentration. We have here the classical prescription for <u>constriction into an arc</u>, and one can expect heavy arcing (with the corresponding damage to electrodes) when α_e approaches 1. This behavior has indeed been observed repeatedly, and has been the focus of a lot of attention, because it limits the practically achievable value of $I^2/\dot{m}$. Since (as we will see) efficiency increases with $I^2/\dot{m}$, this is a major hurdle in the path towards efficient MPD operation. It has been dubbed "the onset condition and we are now in a position to see what it implies quantitatively. Setting (27) to unity, we get

$$\frac{H}{w} \frac{\mu_0 I^2}{\dot{m}} = \sqrt{\frac{9\sqrt{3}}{4} \frac{eV_i'}{m_i}} \tag{28}$$

and from the exit velocity expression (18),

$$u_e = \frac{1}{2}\sqrt{\frac{9\sqrt{3}}{4} \frac{eV_i'}{m_i}} = 0.987\sqrt{\frac{eV_i'}{m_i}} \tag{29}$$

The velocity $\sqrt{\frac{2eV_i'}{m_i}}$ at which the particle's kinetic energy would be capable of ionizing it is called the "Alfvèn critical speed". Many years ago Alfvèn used this conversion of kinetic to ionization energy to construct a model of the "condensation" of matter expanding from the proto-Sun to form the existing planets. We see here that the exhaust speed of an MPD thruster (or a PPT, which works on the same principles) is limited roughly to the Alfvèn critical speed of the gas used (if exit arcs are to be avoided).

For various gases, assuming $V_i' = 2V_i$, we find

Gas	Hydrogen	Nitrogen	Argon	Lithium
M_i (g/mol)	1	14	40	7
V_i (volts)	13.6	14.6	15.8	5.4
$(I_{sp})_{MAX}$ (s)	5,120	1,420	870	1,220

Expressing $m_i = \frac{M}{N_a}$ (N_a=Avogadro's number), and using kA for I and g/s for $\dot{m}$, Equation (28) can also be rewritten as

$$\frac{I^2\sqrt{M}}{\dot{m}}\left(\frac{kA^2\,(g/mol)^{1/2}}{g/s}\right) \simeq 15.4\,\frac{w}{H}\sqrt{V_i'} \qquad (30)$$

volts

For Argon at 6 g/s and w/H=4, this predicts an "onset current" $I^* \simeq 18kA$. Experimental values tend to cluster around $I^* \sim 20 - 23\,kA$, in reasonable agreement. Much of the difference is simply due to the geometry (coaxial vs. rectangular). The scaling of I^* with $\dot{m}^{1/2}$ and with $M^{-1/2}$ is also well documented experimentally.

Efficiency

Accounting only for the power lost to ohmic dissipation and to near-electrode voltage drops ($\Delta V = \Delta V_{cathode} + \Delta V_{anode} \sim 10 - 20$ Volts), we have

$$\eta = \frac{\frac{1}{2}\dot{m}u_e^2}{\frac{1}{2}\dot{m}u_e^2 + D + I\Delta V} \qquad (31)$$

From the equations derived before,

$$\eta = \frac{\frac{1}{2}\dot{m}\left(\frac{1}{2}\frac{H}{w}\frac{\mu_0 I^2}{\dot{m}}\right)^2}{\frac{1}{2}\dot{m}\left(\frac{1}{2}\frac{H}{w}\frac{\mu_0 I^2}{\dot{m}}\right)^2 + \frac{4}{9\sqrt{3}}\left(\frac{H}{w}\right)^2\frac{\mu_0^2 I^4}{\dot{m}} + I\Delta V}$$

$$\eta = \frac{1}{1 + \underbrace{\frac{32}{9\sqrt{3}}}_{3.05} + \frac{2I\Delta V}{\dot{m}\left(\frac{1}{2}\frac{H}{w}\frac{\mu_0 I^2}{\dot{m}}\right)^2}}$$

$$\eta = \frac{1}{3.05 + \frac{8(\Delta V)\dot{m}}{\mu_0^2 I^3}\left(\frac{w}{H}\right)^2} \qquad (32)$$

Even if the voltage drops could be eliminated, this efficiency is no higher than $\eta_{MAX} = \frac{1}{3.05} = 0.328$. The best that can be done in the presence of the voltage drop is to approach "onset", at which point one gets

$$\eta^* = \frac{1}{3.05 + 2.88\Delta V\left(\frac{w/H}{\mu_0 \dot{m}}\right)^{1/2}\left(\frac{m_i}{eV_i'}\right)^{3/4}} \qquad (33)$$

For Argon, $\dot{m} = 6$ g/s, $\frac{w}{H} = 4$, $\Delta V = 10$ Volts this gives $\eta^* = 0.259$.

Values of this order have been reported very often, and it has proven very difficult to exceed $\eta = 0.3$ with Argon at least. Most of the inefficiency is seen to arise from the strong <u>dissipation</u> in the inlet and exit layers. This is intrinsic to the <u>constant area</u> geometry. Although it may not be obvious at this point, a <u>convergent-divergent geometry</u> has the effect of weakening these dissipative layers, and can conceivably be exploited to improve efficiency (and retard "onset"). There is some evidence for this in experiments. The analysis showing the effect of a convergent-divergent geometry (or, in a more limited form, of a purely divergent geometry), can be found in Ref. 1.

References:

Ref. 1: Martinez-Sanchez, M. "Structure of Self-Field Accelerated Plasma Flows." *J. of Propulsion and Plasma* 7, no. 1 (Jan-Feb 1991): 56-64.