

Lecture 10: Electric Propulsion - Some Generalities on Plasma (and Arcjet Engines)

Ionization and Conduction in a High-pressure Plasma

A normal gas at $T \lesssim 3000\text{K}$ is a good electrical insulator, because there are almost no free electrons in it. For pressure $\gtrsim 0.1$ atm, collision among molecule and other particles are frequent enough that we can assume local Thermodynamic Equilibrium, and in particular, ionization-recombination reactions are governed by the Law of Mass Action. Consider neutral atoms (n) which ionize singly to ions (i) and electrons (e):



One form of the Law of Mass Action (in terms of number densities $n_j = \frac{P_j}{kT}$, where T is the same for all species) is

$$\frac{n_e n_i}{n_n} = S(T) \quad (2)$$

Where the "Saha function" S is given (according to Statistical Mechanics) as

$$S(T) = 2 \frac{q_i}{q_n} \left(\frac{2\pi m_e kT}{h^2} \right)^{3/2} e^{-\frac{eV_i}{kT}} \quad (3)$$

- q_i = Ground state degeneracy of ion (= 1 for H^+)
- q_n = Ground state degeneracy of neutral (= 2 for H)
- m_e = mass of electron = 0.91×10^{-30} Kg
- k = Boltzmann constant = 1.38×10^{-23} J/K
(Note: $k = R/\text{Avogadro's number}$)
- h = Plank's constant = 6.62×10^{-34} J.s.
- V_i = Ionization potential of the atom (volts)
($V_i = 13.6$ V for H)

Except for very narrow "sheaths" near walls, plasmas are quasi-neutral:

$$n_e = n_i \quad (4)$$

So that

$$\frac{n_e^2}{n_n} = S(T) \quad (3')$$

can be used.

Given T , this relates n_e to n_n . A second relation is needed and very often it is a specification of the overall pressure

$$P = (n_e + n_i + n_n)kT = (2n_e + n_n)kT \quad (5)$$

Combining (3') and (5),

$$n_e^2 = S(T) \left(\frac{P}{kT} - 2n_e \right) = S(T)(n - 2n_e)$$

Where $n = \frac{P}{kT}$ is the total member density of all particles.

We then have

$$n_e^2 + 2Sn_e - Sn = 0$$

$$n_e = -S + \sqrt{S^2 + Sn} = \frac{n}{1 + \sqrt{1 + \frac{n}{S(T)}}} \quad (6)$$

Since S increases very rapidly with T , the limits of (6) are

$$n_e \xrightarrow{T \rightarrow 0} \sqrt{Sn} \left(\xrightarrow{T \rightarrow 0} 0 \right) \quad (\text{Weak ionization})$$

$$n_e \xrightarrow{T \rightarrow \infty} \frac{n}{2} \quad (\text{Full ionization})$$

Once an electron population exists, an electric field \vec{E} will drive a current density \vec{j} through the plasma. To understand this quantitatively, consider the momentum balance of a "typical" electron. It sees an electrostatic force

$$\vec{F}_E = -e\vec{E} \quad (7)$$

It also sees a "frictional" force due to transfer of momentum each time it collides with some other particle (neutral or ion). Collisions with other electrons are not counted, because the momentum transfer is in that case internal to the electron species. The ions and neutrals are almost at rest compared to the fast-moving electrons, and we define an effective collision as one in which the electron's directed momentum is fully given up. Suppose there are ν_e of these collisions per second (ν_e = collision frequency per electron). The electron loses momentum at a rate $-m_e \bar{V}_e \nu_e$, where \bar{V}_e = mean directed velocity of electrons, and so

$$\vec{F}_{\text{Friction}} = -m_e \bar{V}_e \nu_e \quad (8)$$

On average,

$$\vec{F}_E + \vec{F}_{\text{Friction}} = 0,$$

or

$$m_e \bar{V}_e v_e = -e \bar{E}$$

$$\bar{V}_e = - \left(\frac{e}{m_e v_e} \right) \bar{E} \quad (9)$$

The group $\mu_e = \frac{e}{m_e v_e}$ is the electron "mobility" ((m/s)/ (volt/m)). The current density is the flux of charge due to motion of all charges. If only the electron motion is counted (it dominates in this case)

$$\vec{j} \equiv -en_e \bar{V}_e \quad (10)$$

and from (9),

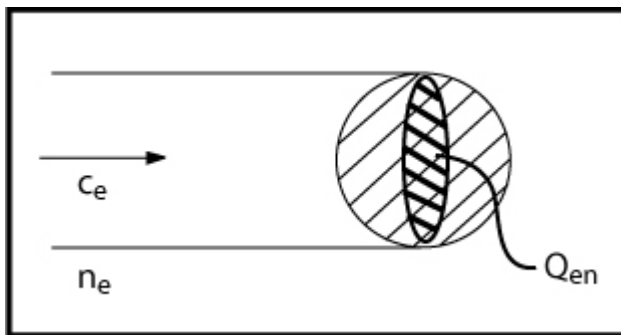
$$\vec{j} = \left(\frac{e^2 n_e}{m_e v_e} \right) \bar{E} \quad (11)$$

The group

$$\sigma = \frac{e^2 n_e}{m_e v_e} \quad (12)$$

is the conductivity of the plasma (Si/m).

Let us consider the collision frequency. Suppose a neutral is regarded as a sphere with a cross-section area Q_{en} .



Electrons moving at random with (thermal) velocity c_e intercept the area Q_{en} at a rate equal to their flux $n_e c_e Q_{en}$. Since a whole range of speeds \bar{c}_e exists, we use the average value \bar{c}_e for all electrons. But this is for all electrons colliding with one neutral. We are interested in the reverse (all neutrals, one electron), so the part of v_e due to neutrals

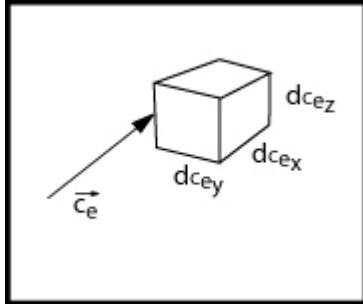
should be $n_n \bar{c}_e Q_{en}$. Adding the part due to ions,

$$v_e = n_n \bar{c}_e Q_{en} + n_i \bar{c}_e Q_{ei} \quad (13)$$

NOTE:

\bar{c}_e is very different (usually much larger) than V_e . Most of the thermal motion is fast, but in random directions, so that on average it nearly cancels out. The non-cancelling remainder is V_e . Think of a swarm of bees moving furiously to and fro, but moving (as a swarm) slowly.

The number of electrons per unit volume that have a velocity vector \vec{c}_e ending in a "box"



$dc_{ex} dc_{ey} dc_{ez} \equiv d^3c_e$ in velocity space is defined as

$$f_e(\vec{c}_e) d^3c_e$$

Where $f_e(\vec{c}_e, \vec{x})$ is the Distribution function of the electrons which depends (for a given location \vec{x} and time t) on the three components of \vec{c}_e . In an equilibrium

situation all directions are equally likely, so $f_e = f_e(|\vec{c}_e|) = f_e(c_e)$ only, and one can show that the form is Maxwellian.

$$f_e = n_e \left(\frac{m_e}{2\pi kT_e} \right)^{3/2} e^{-\frac{m_e c_e^2}{2kT_e}} \quad ; \quad (c_e^2 = c_{ex}^2 + c_{ey}^2 + c_{ez}^2) \quad (14)$$

With the normalization

$$\iiint f_e d^3c_e = n_e.$$

The mean velocity is then

$$\bar{c}_e \equiv \iiint c_e f_e d^3c_e$$

and direct calculation gives

$$\bar{c}_e = \sqrt{\frac{8 kT_e}{\pi m_e}} \quad (15)$$

For Hydrogen atoms,

$$\bar{c}_e = 6210 \sqrt{T_e} \quad (\text{m/s, with } T_e \text{ in K}) \quad (16)$$

NOTE:

If there is current, the distribution cannot be strictly Maxwellian (or even isotropic). But since $V_e \ll c_e$, the mean thermal velocity is very close to Equation (15) anyway.

Regarding the cross sections Q_{en} , Q_{ei} , they depend on the collision velocity \bar{c}_e , especially Q_{ei} . This is because the e-i coulombic interaction is "soft", so a very fast electron can pass nearly undeflected near an ion, whereas a slow one will be strongly deflected. The complete theory yields an expression

$$Q_{ei} \approx 2.95 \times 10^{-10} \frac{\ln \Lambda}{T} \quad (\text{m}^2) \quad (17)$$

(T in Kelvin)

where

$$\ln \Lambda \approx -11.35 + 2 \ln T(\text{K}) - \frac{1}{2} \ln P(\text{atm}) \quad (18)$$

so that $\ln \Lambda$ is usually around 6-12, and can even be taken as a constant (~ 8) in rough calculations. For the neutral hydrogen atoms, the collisions are fairly "hard", and one can use the approximation

$$Q_{eH} \approx 2 \times 10^{-19} \text{m}^2 \quad (19)$$

Numerical Example

Consider Hydrogen at $P=1$ atm. For $T \geq 4000\text{K}$, diatomic H_2 is not present anymore ($\text{H}_2 \rightarrow 2\text{H}$). So ionization is from atomic hydrogen, H, for which

$$V_i = 13.6 \text{ volts,}$$

$$q_i = 1,$$

$$q_n = 2,$$

so that

$$S = 2.42 \times 10^{21} T^{3/2} e^{-\frac{157,800}{T}} \quad (\text{m}^{-3}).$$

We also find

$$n = \frac{7.34 \times 10^{27}}{T} \quad (\text{m}^{-3}),$$

$$\bar{c}_e = 6210\sqrt{T} \quad (\text{m/s}),$$

$$\sigma = 2.821 \times 10^{-8} n_e / v_e$$

$$(v_e = v_{ei} + v_{en}).$$

The results are shown below.

T(K)	5,000	6,000	7,000	8,000	10,000	12,000	14,000	16,000
$n(\text{m}^{-3})$	1.47×10^{24}	1.22×10^{24}	1.048×10^{24}	9.18×10^{23}	7.34×10^{23}	6.12×10^{23}	5.24×10^{23}	4.59×10^{23}
$S(\text{m}^{-3})$	1.68×10^{13}	4.26×10^{15}	2.30×10^{17}	4.70×10^{18}	3.393×10^{20}	6.189×10^{21}	5.104×10^{22}	2.55×10^{23}
$n_e(\text{m}^{-3})$	4.97×10^{18}	7.216×10^{19}	4.906×10^{20}	2.072×10^{21}	1.545×10^{22}	5.565×10^{22}	1.203×10^{23}	1.716×10^{23}
α	3.4×10^{-6}	5.9×10^{-5}	4.7×10^{-4}	0.0023	0.0215	0.1000	0.298	0.597
$\bar{c}_e(\text{m/s})$	4.39×10^5	4.81×10^5	5.20×10^5	5.56×10^5	6.21×10^5	6.80×10^5	7.35×10^5	7.86×10^5
$\ln \Lambda$	5.68	6.05	6.36	6.62	7.07	7.44	7.74	8.01
$v_{ei}(\text{s}^{-1})$	1.46×10^8	1.72×10^9	9.76×10^9	3.51×10^{10}	2.00×10^{11}	5.77×10^{11}	1.030×10^{12}	1.244×10^{12}
$v_{en}(\text{s}^{-1})$	1.29×10^{11}	1.15×10^{11}	1.09×10^{11}	1.02×10^{11}	8.73×10^{10}	6.81×10^{10}	4.16×10^{10}	1.82×10^{10}
$v_e(\text{s}^{-1})$	1.29×10^{11}	1.17×10^{11}	1.19×10^{11}	1.37×10^{11}	2.87×10^{11}	6.45×10^{11}	1.072×10^{12}	1.262×10^{12}
$\sigma(\text{Si/m})$	1.09	17.4	116.3	426.7	1519	2434	3166	3836

Notice

(a) Coulomb-dominated ($v_{ei} \gg v_{en}$) for $T \gtrsim 8000 \text{ K}$

(b) Rapid rise of ionization fraction α for $7000 \lesssim T \lesssim 10,000 \text{ K}$

(c) Conductivity above

100 Si/m	for $T \gtrsim 7000 \text{ K}$
1,000 Si/m	$T \gtrsim 9000 \text{ K}$

Ohmic Dissipation - Stability, constriction

The conductivity σ increases rapidly with T in the fully Coulomb-dominated range,

$$\sigma = \frac{0.0153 T_e^{3/2}}{\ln \Lambda} \quad (T_e \text{ in K, } \sigma \text{ in Si/m}).$$

Notice also how, in this limit (which occurs at high temperature, as α approaches 1), the conductivity becomes independent of the kind of gas in question, except for small influences hidden in $\ln \Lambda$.

One important consequence of $\sigma = \sigma(T)$ is the tendency for current to concentrate into “filaments”, or “arcs”. To understand this, consider the amount of work done by electric forces to overcome the “friction” on the electrons due to collisions. The force on the n_e electrons in a unit volume is $-en_e\vec{E}$, and these electrons reach a terminal velocity \vec{V}_e as they slide against friction. Hence the power dissipated per unit volume is

$$D_{OH} = -en_e\vec{E}\cdot\vec{V}_e = -(en_e\vec{V}_e)\cdot\vec{E}$$

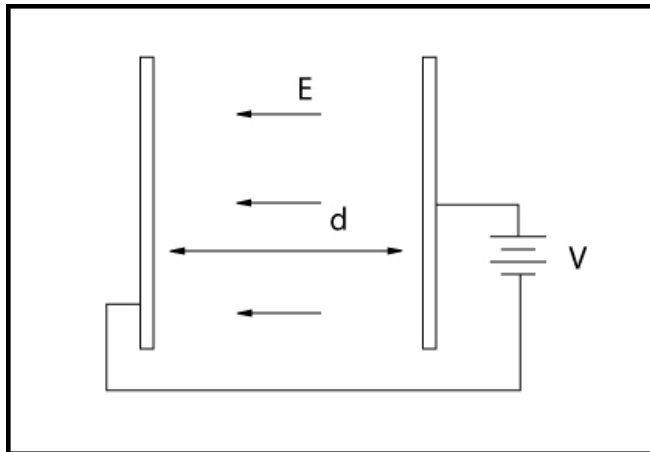
$$D_{OH} = \vec{j}\cdot\vec{E} \quad (\text{Ohmic dissipation}) \quad (20)$$

Since Ohm’s law gives $\vec{j} = \sigma\vec{E}$, we can put

$$D_{OH} = \sigma E^2 \quad \text{or} \quad D_{OH} = \frac{j^2}{\sigma} \quad (21)$$

The simplest situation is one with an initially uniform plasma subject to a constant applied field \vec{E} , such as would occur between the plates of a plane capacitor:

Regardless of the path taken by the current, if the plates are large and the gap is small, the field $E = \frac{V}{d}$ remains unchanged. If we now look at



$$D_{OH} = \sigma E^2 ,$$

we see that the dissipation becomes large wherever the conductivity (hence the temperature) is large. Starting from uniform temperature, if a small non-uniformity arises such that T is higher along a certain path, that path becomes more conductive, heats up due to extra Ohmic dissipation, and this reinforces the initial nonuniformity. The result is a constriction of the current into a filament or “arc”.

In principle, the constriction process would continue indefinitely and lead to arcs of zero radius and infinite current density. But as the temperature profile steepens, heat will increasingly diffuse away from the hot core to the cooler surroundings, and, provided it can be removed efficiently from there, an equilibrium is eventually reached at some finite arc radius and arc core temperature. Clearly, the detailed end result will depend on the details of the thermal management of the gas: the more efficient the cooling of the background, the more the constriction can progress, and the hotter the eventual arc. This counter-intuitive result (more cooling leads to hotter arcs) is one of several paradoxical properties of arcs, all of them related to their being the result of a statically unstable situation. We analyze this behavior next.

Sample Physical Properties of High-temperature Gases (Near Equilibrium)

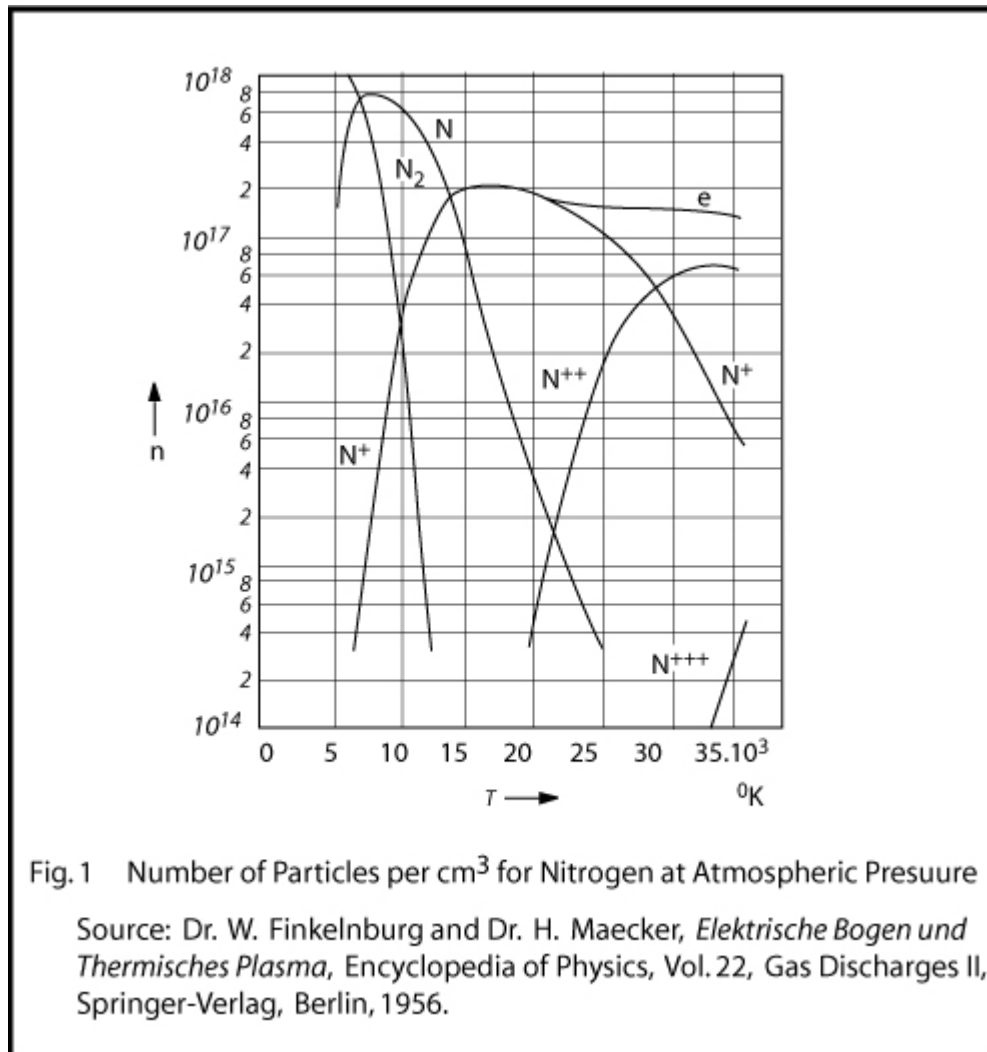


Figure 1: Equilibrium Composition of Nitrogen at P=1 atm.

Note: N₂ replaced by N at ~ 7,000K, then by N⁺ at ~ 14,000K and then by N⁺⁺ at ~ 29,000K. Electron density satisfies $n_e = n^+ + 2n^{++}$

For Hydrogen, similar, but all transitions at lower temperature (and, of course, there is no H⁺⁺)

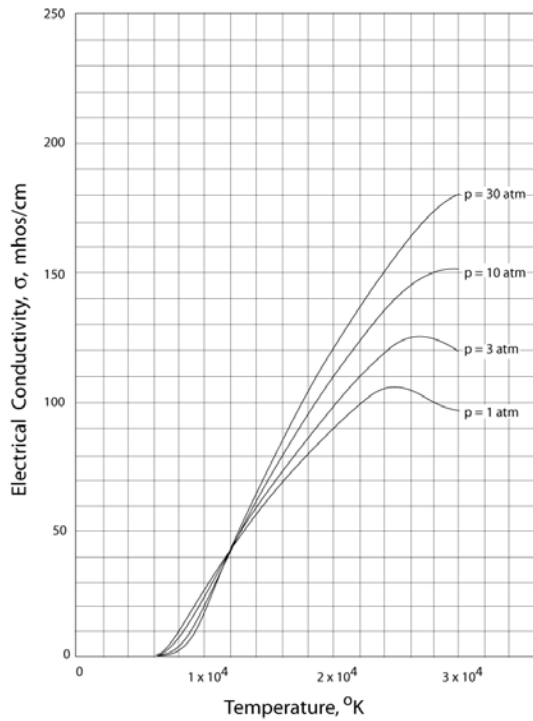


Fig.2 Electrical Conductivity of Nitrogen Versus Temperature

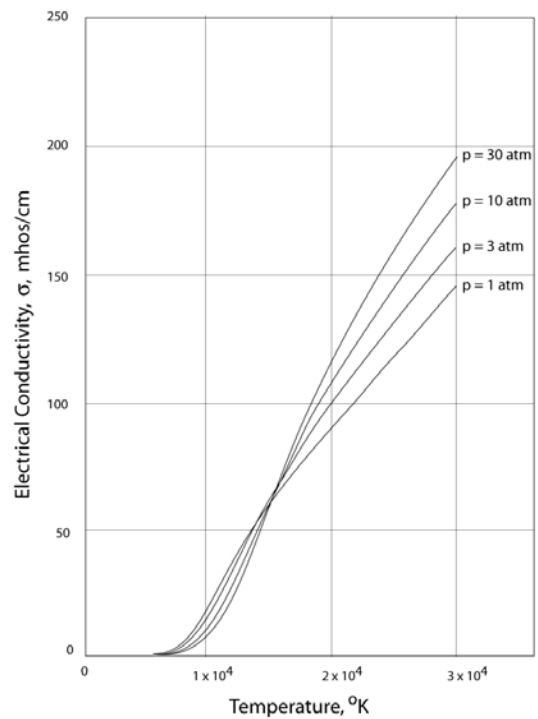


Fig.3 Electrical Conductivity of Hydrogen Versus Temperature

Source: J.Yos, *Transport Properties of Nitrogen, Hydrogen, Oxygen, and Air to 30,000° K*, AVCO, Technical Memorandum RAD-TM-63-7, 1963, reprint-1966.

Figure 2, 3: Electrical Conductivity of Nitrogen and Oxygen

Note: Weak dependence on

gas type
Pressure

Units are mho/cm \equiv Si/cm.

1 Si/cm = 100 Si/m

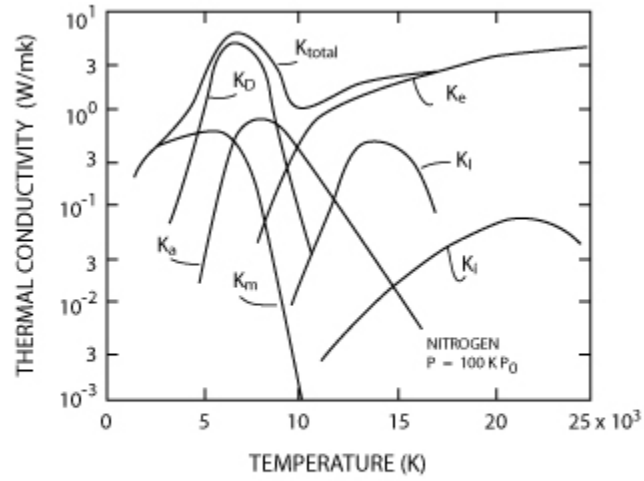


Fig. 4 Individual contributions to the heat conductivity of a nitrogen plasma (Burhorn, 1959). Contribution due to molecules K_m , atoms K_{at} , electrons K_e , ions K_i , dissociation K_D , ionization K_i .

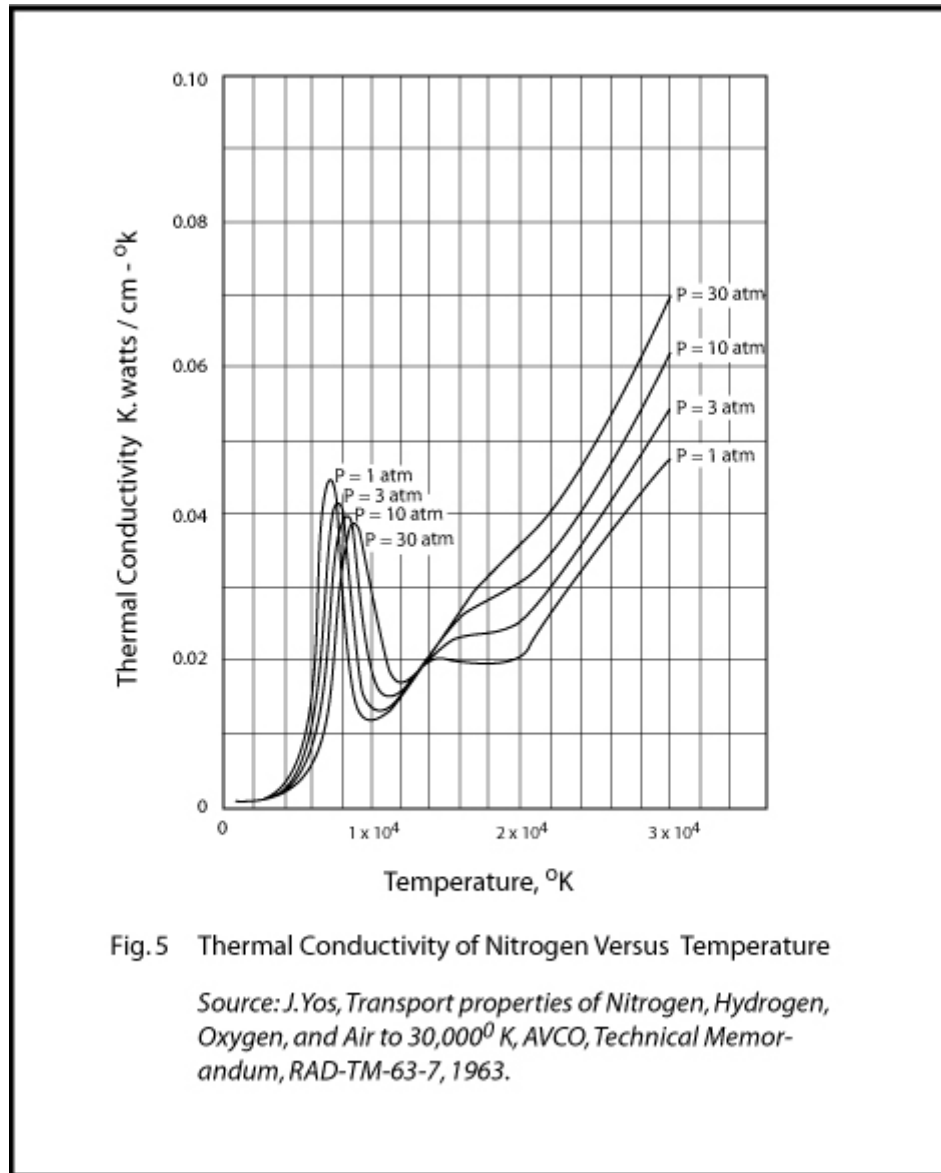


Figure 4, 5: Thermal Conductivity of Nitrogen

Note: The molecular contribution K_m is due to the transport of energy by the random thermal motion of N_2 molecules. It increases weakly with T when there are molecules, but, of course it disappears when they dissociate ($T \geq 7000K$). Similar physics applies to the atoms contribution K_a , the ion contribution K_i and the electron contribution, K_e (these rise rapidly at first, when these species first appear, then drop rapidly out when they in turn dissociate).

The most striking new feature are K_D (the "dissociation contribution"), and K_I (the "ionization contribution"). These are akin to heat-pipe effects, and they appear in temperature ranges where dissociation (or ionization) are very sensitive to T (around 7000K for K_D , around 15,000K for K_I). The mechanism is as follows: When there is a temperature gradient, molecules are dissociating at a higher rate

than they are forming by recombination in high-T sections, while the reverse happens in low-T parts. The dissociation products are continuously diffusing from hot to cold regions. When they are created (high T) they absorb the heat of dissociation (very large) and they deliver it when they recombine (at low T). This creates a net (strong) heat transport from high to low T, and hence a thermal conductivity K_D . The same description applies to K_I , except the heat transported is now the ionization energy. Note the very strong K_D impact in the natural scale Figure (5).

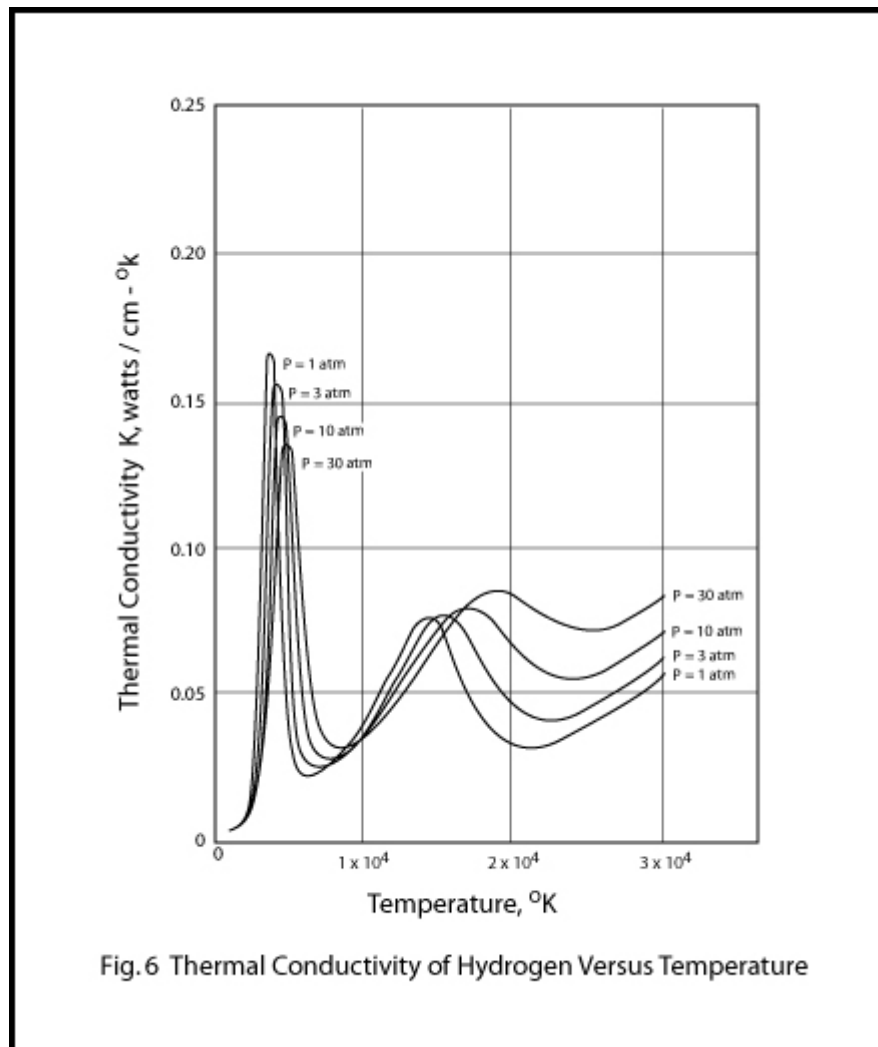


Fig.6 Thermal Conductivity of Hydrogen Versus Temperature

Figure 6: Thermal Conductivity of Hydrogen

Qualitatively similar to Nitrogen, the K_I component is here even more evident (around $16,000$ K).