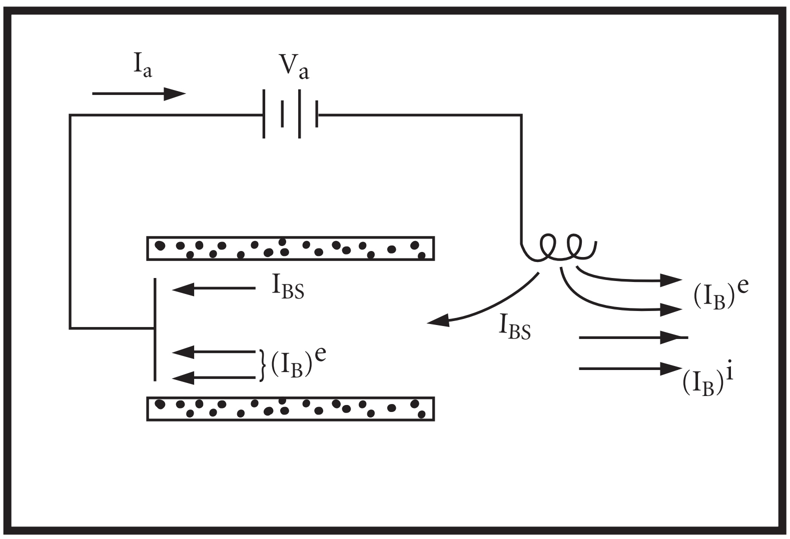
Lecture 18: Hall Thruster Efficiency

For a given mass flow $\dot{m}$ and thrust F, we would like to minimize the running power P. Define a thruster efficiency

$$\eta = \frac{\left(F^2 / 2\dot{m}\right)}{P} \quad (1)$$

where $\left(F^2 / 2\dot{m}\right)$ is the minimum required power. The actual power is

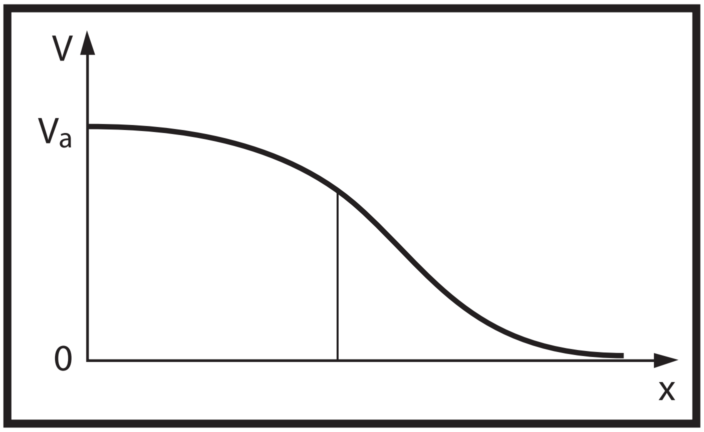
$$P = I_a V_a \quad (2)$$



Where V_a is the accelerating voltage and I_a the current through the power supply (or anode current, or also cathode current). Of the I_a current of electrons injected by the cathode, a fraction I_B goes to neutralize the beam, and the rest, I_{BS} back-streams into the thruster.

Since no <u>net</u> current is lost to the walls,

$$I_a = I_B + I_{BS} \quad (3)$$



The thrust is due to the accelerated ions only. These are created at locations along the thruster which have different potentials V(x), and hence accelerate to different speeds. Then

$$F = \int c\, d\dot{m}_i \tag{4}$$

where

$$c = \sqrt{\frac{2eV}{m_i}} \tag{5}$$

Suppose the part $d\dot{m}_i$ of $\dot{m}_i$ is created in the region where V decreases by dV, and define an "ionization distribution function" f(V) by

$$\left(\frac{d\dot{m}_i}{\dot{m}_i}\right) = -f\left(\frac{V}{V_a}\right)\left(\frac{dV}{V_a}\right) \tag{6}$$

or, with

$$\frac{V}{V_a} = \varphi,$$

$$\frac{d\dot{m}_i}{\dot{m}_i} = -f(\varphi)\, d\varphi.$$

From the definition, $f(\varphi)$ satisfies

$$\int_0^1 f(\varphi)\, d\varphi = 1 \tag{7}$$

Then, from (4) and (5),

$$F = \dot{m}_i \sqrt{\frac{2eV_a}{m_i}} \int_0^1 \sqrt{\varphi}\, f(\varphi)\, d\varphi \tag{8}$$

and hence the efficiency is

$$\eta = \frac{\left(\dot{m}_i\right)^2 \frac{2eV_a}{m_i} \left(\int_0^1 \sqrt{\varphi}\, f(\varphi)\, d\varphi\right)^2}{2\dot{m} V_a I_a} \tag{9}$$

Notice that the beam current I_B is related to $\dot{m}_i$ by $I_B = \frac{e}{\dot{m}_i}\dot{m}_i$. We can therefore re-write (9) as

$$\eta = \left(\frac{\dot{m}_i}{\dot{m}}\right)\left(\frac{I_B}{I_a}\right)\left(\int_0^1 \sqrt{\varphi}\, f(\varphi)\, d\varphi\right)^2 \tag{10}$$

where each of the factors is less than unity and can be assigned a separate meaning:

(11) $\frac{\dot{m}_i}{\dot{m}} \equiv \eta_u$ is the "utilization factor", i.e., it penalizes neutral gas flow.

(12) $\frac{I_B}{I_a} = \eta_a$, the "backstreaming efficiency" penalizes electron backstreaming.

(13) $\left(\int_0^1 \sqrt{\varphi}\, f(\varphi)\, d\varphi\right)^2 = \eta_E$, the "nonuniformity factor" is less than unity because of the nonuniform ion velocity

It is clear that, since $\int_0^1 f(\varphi)\, d\varphi = 1$, we want to put most of $f(\varphi)$ where $\sqrt{\varphi}$ is greatest, namely, we want to produce most of the ionization near the inlet. In that case $f(\varphi) = \delta(\varphi - 1)$, and $\eta_E = 1$. A somewhat pesimistic scenario would be $f(\varphi) = 1$, namely $\frac{d\dot{m}_i}{dx}$ proportional to $-\frac{dV}{dx}$, i.e., ionization rate proportional to field strength.

In that case

$$\eta_E = \left(\int_0^1 \sqrt{\varphi} \times 1 \times d\varphi\right)^2 = \left(\frac{2}{3}\right)^2 = \frac{4}{9}$$

Measurements[1] tend to indicate $\eta_E \sim 0.6 - 0.9$, which means that ionization <u>tends</u> to occur early in the channel. This is to be expected, because that is where the backstreaming electrons have had the most chance to gain energy by "falling" up the potential.

The factor $\eta_u = \dfrac{\dot{m}_i}{\dot{m}}$ is related to the ionization fraction. Putting

$$\dot{m}_i = n_e c_i A \quad , \quad \dot{m} = \dot{m}_i + \dot{m}_n = \left(n_e c_i + n_n c_n\right) A \, ,$$

$$\eta_u = \frac{\left(n_e/n_n\right)\left(c_i/c_n\right)}{1+\left(n_e/n_n\right)\left(c_i/c_n\right)} \qquad (11)$$

Since c_i/c_n is large (c_n ~ neutral speed of sound, i.e., a few hundred m/sec, while $c_i \simeq gI_{sp} \sim 20{,}000\text{m/sec}$), η_u can be high even with n_e/n_n no more than a few percent. Data[1] show η_u ranging from 40% to 90%.

The factor $\eta_E = I_B/I_a$ requires some discussion. Most of the ionization is due to the backstreaming electrons, so that we are not really free to drive I_B towards $I_a\left(I_{BS} = I_a - I_B\right)$. What we need to strive for is

(a) Conditions which favor creation of as many ions as possible per backstreaming electron, and

(b) Minimization of ion-electron losses to the walls, once they are created.

This can be quantified as follows: Let β be the number of secondary electrons (and of ions) produced per backstreaming electron, and let α be the fraction of these new $e-i$ pairs which is lost by recombination on walls. Then, per backstreaming electron, $\left(1-\alpha\right)\beta$ ions make it to the beam, and an equal number of cathode electrons are used to neutralize them. Therefore

$$\eta_a = \frac{I_B}{I_a} = \frac{\left(1-\alpha\right)\beta}{1+\left(1-\alpha\right)\beta} \qquad (12)$$

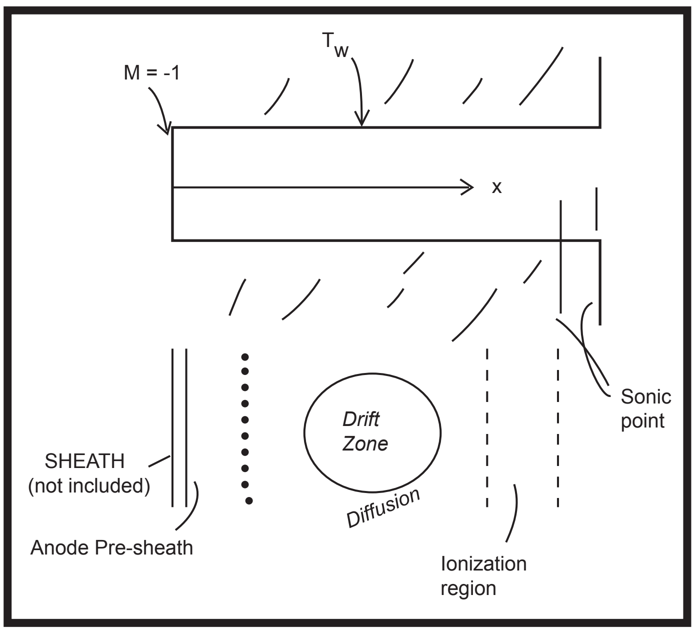
Clearly, we want $\beta >> 1$ and $\alpha << 1$. The first $\left(\beta >> 1\right)$ implies lengthening the electron path by means of the applied radial magnetic field, and also using accelerating potentials which are not too far from 5/2 times the range of energies where ionization is most efficient (typically 30-80 Volts). This last condition creates some difficulties with heavy ions, which require higher accelerating potentials for a given exit speed.

The condition $\alpha << 1$ implies minimization of insulation surfaces on which the recombination can take place, and arrangement of the electric fields such that ions are not directly accelerated into walls. This is difficult to achieve without detailed surveys of equipotential surfaces.

Reference:

Komurasaki, K, Hirakowa, M. and Arakawa, Y., IEPC paper 91-078. 22nd Electric Propulsion Conference, Viareggio, Italy, Oct. 1991.

1-D Model of Hall Thruster



Define

$$\Gamma_e = n_e v_e \qquad (v_e < 0, \text{ so } \Gamma_e < 0) \tag{1}$$

$$\Gamma_i = n_e v_i \qquad (+ \text{ or } -) \tag{2}$$

$$\Gamma_n = n_n v_n \qquad v_n \simeq \text{constant} \simeq \sqrt{\frac{5}{3}\frac{kT_w}{m_i}} \tag{3}$$

1. Conservation of particles

$$\frac{d\Gamma_e}{dx} = \frac{d\Gamma_i}{dx} = -\frac{d\Gamma_n}{dx} = n_e \nu_{ion} \tag{4}$$

Where

$$\nu_{ion} = n_n R_i(T_e) \quad (5)$$

$$R_i = \sigma_0 \bar{c}_e \left(1 + 2\frac{kT_e}{E_i}\right) e^{-\frac{E_i}{kT_e}} \quad (6)$$

$$\bar{c}_e = \sqrt{\frac{8}{\pi}\frac{kT_e}{m_i}} \quad (7)$$

and, for X_e, $\sigma_0 \simeq 3.6 \times 10^{-20} m^2$, $E_i = eV_i$, $V_i = 12.1V$; $m_i = 2.2 \times 10^{-25}$ kg

From (4), two first integrals :

$$\Gamma_i - \Gamma_e = \Gamma_d = \text{constant} \quad (8)$$

$$\Gamma_i + \Gamma_n = \Gamma_m = \text{constant} \quad (9)$$

and

$$\Gamma_d = \frac{I_a}{Ae}, \Gamma_m = \frac{\dot{m}}{Am_i} \quad (10)$$

(8)–(10) → Conservation Equations

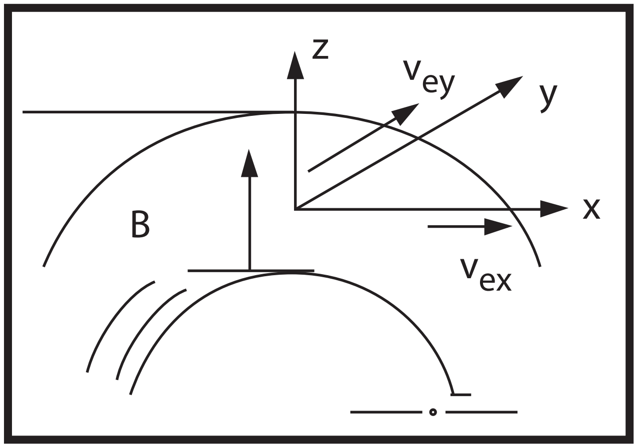
2. Ion Momentum Equation

The force per unit volume on the ion gas is eEn_e (from the axial electric field E). There is also a "pick-up" drag due to ionization. For each ionization event, a new ion is "incorporated" to the ion population (of velocity v_i), jumping from the neutral velocity v_n; this gives a drag $-n_e \nu_{ion} m_i (v_i - v_n)$. We then have

$$m_i v_i \frac{dv_i}{dx} = eE - m_i \nu_{ion}(v_i - v_n) \quad (11)$$

3. Electron Momentum Equation

Consider first only classical electron collisions (say, with neutrals). The vector equation of motion of electrons, including electric force, magnetic force and collisional "drag" (and neglecting inertia) is



$$\nabla P_e = -en_e\left(\vec{E} + \vec{v}_e \times \vec{B}\right) - m_e \nu_{en} n_e \vec{v}_e \qquad (12)$$

$$\left(\nu_{en} = n_n \bar{c}_e \sigma_{en}\right)$$

Project on x, y:

$$\left(E_y = 0,\ \frac{\partial P_e}{\partial x} = 0\right) \longrightarrow \begin{cases} \dfrac{dP_e}{dx} = -en_e\left(E_x + v_{ey}B\right) - m_e \nu_{en} n_e v_{ex} & (13) \\ 0 = -en_e\left(0 - v_{ex}B\right) - m_e \nu_{en} n_e v_{ey} & (14) \end{cases}$$

From (14),

$$v_{ey} = \frac{eB}{m_e \nu_{en}} v_{ex} = \frac{\omega_c}{\nu_{en}} v_{ex} \qquad (15)$$

Substitute in (13):

$$\frac{dP_e}{dx} = -en_eE_x - en_eB\frac{\omega_c}{\nu_{en}} v_{ex} - m_e \nu_{en} n_e v_{ex}$$

$$= -en_eE_x - m_e n_e v_{ex}\left(\frac{\omega_c^2}{\nu_{en}} + \nu_{en}\right) \qquad (16)$$

In the Hall thruster plasma, $\nu_{en} << \omega_c$ (low collisionality), so second term in parenthesis is neglected. The quantity $\frac{\omega_c^2}{\nu_{en}}$ acts then as an “effective collision frequency”, accounting for the magnetic effect

$$"\nu_e" = \frac{\omega_c^2}{\nu_{en}} \qquad (17)$$

If there were several types of real collisions, such as $e-n_1$ and $e-n_2$, then $\nu_{en}=\nu_{en_1}+\nu_{en_2}$ in (17). We write the momentum equation (with $P_e = n_e kT_e$) as

$$\boxed{\frac{d}{dx}\left(n_e kT_e\right) = -en_eE_x - m_e \underbrace{n_e v_e}_{\Gamma_e} \nu_e} \qquad \left(v_e \equiv v_{ex}\right) \tag{18}$$

In the SPT type of Hall thruster, the dominant scattering affect is actually not collisions, but the random deflections due to plasma turbulence ("anomalous diffusion, Bohm diffusion"). To see what modifications this introduces, consider a simple case with $T_e=$ constant and $E_x=0$. Equation (18) then gives

$$\Gamma_e = -\frac{kT_e}{m_e\nu_e}\frac{dn_e}{dx} \tag{19}$$

So that the diffusivity is $D=\dfrac{kT_e}{m_e\nu_e}$. Using (17), with more than one type of collision,

$$D = \frac{kT_e}{m_e\omega_c^2}\left(\nu_{en_1}+\nu_{en_2}\right) \tag{20}$$

If anomalous (Bohm) diffusion dominates, it is known empirically that

$$D = D_{Bohm} \simeq \alpha_B\frac{kT_e}{eB}\left(\alpha_B \simeq \frac{1}{16}\right).$$

If we liken this to collisionality effect, say, ν_{en_2}, then

$$\alpha_B\frac{kT_e}{eB} = \frac{kT_e}{me\omega_c^2}\nu_{en_2}\text{, or}$$

$$\boxed{\nu_{en_2} = \alpha_B\omega_c} \tag{21}$$

Therefore, if we want to account for both, classical e-n collisions <u>and</u> Bohm diffusion effects, we will define "ν_e" as (from (17)

$$\boxed{\nu_e = \frac{\omega_c^2}{\nu_{en}+\alpha_B\omega_c}} \tag{22}$$

4. **<u>Electron Energy Equation</u>**

The convected enthalpy flux of the electron gas is $\Gamma_e\dfrac{5}{2}kT_e$. Its divergence is the net rate of work done on this gas per unit volume, minus the work rate expended in ionization and excitation of neutrals:

$$\boxed{\frac{d}{dx}\left(\Gamma_e\frac{5}{2}kT_e\right) = -e\Gamma_eE_x - n_e\nu_{ion}E_i'} \tag{23}$$

Where $E_i^{'}$ is roughly 2-3 times the actual ionization energy E_i to include the radiative losses due to excitation by electron impact, followed by prompt photon emission. Notice that, since $n_e \nu_{ion} = \frac{d\Gamma_e}{dx}$ this can also be written as

$$\boxed{\frac{d}{dx}\left[\Gamma_e\left(\frac{5}{2}kT_e + E_i^{'}\right)\right] = -e\Gamma_e E_x} \tag{24}$$

5. **Solving for Derivatives**

We combine here for clarity, the main equations:

$$\frac{d\Gamma_e}{dx} = \frac{d\Gamma_i}{dx} = -\frac{d\Gamma_x}{dx} = \nu_{ion} n_e \tag{25}$$

$$m_i v_i \frac{dv_i}{dx} = eE_x - m_i \nu_{ion}\left(v_i - v_n\right) \tag{26}$$

$$\frac{d\left(n_e kT_e\right)}{dx} = -en_e E_x - m_e \Gamma_e \nu_e \tag{27}$$

$$\frac{d}{dx}\left(\frac{5}{2}kT_e\Gamma_e\right) = -e\Gamma_e E_x - n_e \nu_{ion} E_i^{'} \tag{28}$$

It is just a matter of algebra to solve for each of the gradients, separately (including the potential gradient $-E_x = \frac{\partial \phi}{\partial x}$).

The results are

$$\left(\frac{5}{3}kT_e - m_i v_i^2\right)\frac{dv_i}{dx} = \frac{5}{3}m_e \nu_e v_e v_i + \nu_{ion}\left[\frac{5}{3}kT_e + m_i v_i\left(v_i - v_n\right) - \frac{v_i}{v_e}\frac{2E_i^{'} + 5kT_e}{3}\right] \tag{29}$$

$$\left(\frac{5}{3}kT_e - m_i v_i^2\right)\frac{dn_e}{dx} = \frac{5}{3}m_e n_e \nu_e v_e - n_e \nu_{ion}\left[m_i\left(2v_i - v_n\right) - \frac{2E_i^{'} + 5kT_e}{3v_e}\right] \tag{30}$$

$$\left(\frac{5}{3}kT_e - m_i v_i^2\right)k\frac{dT_e}{dx} = -\frac{2}{3}m_e \nu_e v_e m_i v_i^2 - m_i \nu_{ion}\left[\frac{2}{3}kT_e\left(2v_i - v_n\right) - \frac{m_i v_i^2 - kT_e}{m_i v_e}\frac{2E_i^{'} + 5kT_e}{3}\right] \tag{31}$$

$$\left(\frac{5}{3}kT_e - m_i v_i^2\right)eE_x = \frac{5}{3}m_e \nu_e v_e m_i v_i^2 + m_i \nu_{ion}\left[\frac{5}{3}kT_e\left(2v_i - v_n\right) - \frac{v_i^2}{v_e}\frac{2E_i^{'} + 5kT_e}{3}\right] \tag{32}$$

Here

$v_e = v_{ex} = \frac{\Gamma_e}{n_e}$ (generally negative).

The most important feature of these equations is the factor $\frac{5}{2}kT_e - m_i v_i^2$ which would appear in the denominator. It becomes zero when

$$v_i = v_{is} = \sqrt{\frac{5}{3}\frac{kT_e}{m_i}} \tag{33}$$

which is the ion-sonic wave speed, an acoustic wave in which both, ions and electrons, undergo compressions and expansions; they are coupled to each other electrostatically, and since electrons are hotter, they provide the "restoring force" kT_e, while the ions, more massive, provide the inertia, m_i.

Because of this, the gas can accelerate across this speed (of the order of 3000-4000 m/s in Xenon) in one of two modes:

(a) Smoothly, if the right-hand sides of all of Equations (29-32) are zero when $v_i = v_{is}$ (actually, if one of them is zero, the others will also be, at $v_i = v_{is}$). This imposes an internal condition on the differential equations, to supplement the boundary conditions. The difficulty is that one does not know a-priory where (in x) this condition will occur. It is also difficult to integrate numerical through this point, because each derivative is of the $\frac{0}{0}$ form. One needs to use L' Hospital's rule to extract the finite ratio (two values normally).

(b) Abruptly, if the right-hand sides are nonzero when $v_i = v_{is}$. In this case, the derivatives (including E_x) are locally infinite, although one can show that they behave as $1/\sqrt{|x - x_s|}$, and so this infinity is integrable. This can only happen at the open end of the channel, just as with a normal open gas pipe discharging into a vacuum. In this case, we impose the end condition $v_i = v_{is}(T_e)$.

Notice that condition (b) can also occur at the inlet (x=0). Infact , it does occur. This is because the anode will develop a negative sheath (electron repelling) in order to restrict the electron capture to the required I_a level. This same sheath will then attract ions, which will therefore enter it at their sonic velocity (a form of Bohm's sheath-edge criterion):

$$v_i(x = 0) = -\sqrt{\frac{3}{5}\frac{kT_e(0)}{m_i}} \tag{34}$$

6. **Boundary Conditions**

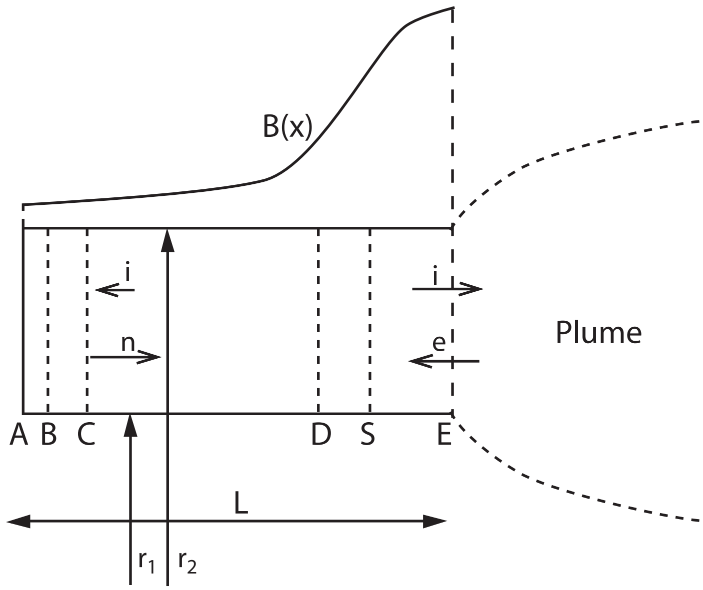
So, in this device we have two sonic points, one (reversed) at inlet, and one (forward) either at the exit plane or somewhere in the channel. This provides either two boundary conditions, or one (Equation (34)) plus one internal condition of smooth sonic passage. Looking at Equations (25-28) we count 6 differential equations. However, we can assume the integration constants Γ_m and Γ_d are prescribed, which leaves as in need of two additional boundary conditions.

[Comment: Prescribing both, Γ_m and Γ_d becomes redundant once full ionization is reached; a more realistic prescription would be Γ_m and the total applied voltage V. However, most of V develops in the supersonic near-plume, outside the channel, and is not captured in this analysis.]

The two missing conditions must play the role of connectors to the outside plume. In an ordinary gas flow, conditions upstream of the sonic point would be fully decoupled from those downstream. But in this problem we are dealing with two counter flowing streams, and electrons, in particular, do carry information from the outside back to the plasma in the channel. The most obvious part is that the mean energy (T_e) of the backstreaming electrons entering the channel must increase (linearly?) as the total voltage V increases. We therefore prescribe $T_e(L)$ as a kind of proxy for the applied voltage. A more subtle effect is that the fraction of the electrons emitted by the downstream cathode which do backstream to the channel must depend on details of the outside region comprising the cathode itself and the near plume. Pending a model of this region, we also prescribe $\frac{\Gamma_e(L)}{\Gamma_d}$.

7. **Anatomy of the Discharge**

Since analytical full solution is out of the question, it is useful to analyze the different regions in the thruster according to the dominant mechanisms in them. Numerical integration is, of course possible, and detailed results are presented in Reference (1) (for the choked-exit case) and Reference (2) (for the smooth sonic passage case). Unsteady effects have also been considered in Reference (3). Figure 1, from Reference (1) shows a useful categorization,



Sketch of the channel: L is the channel length; r_1 and r_2 are the inner and outer radius, respectively; $_n$, $_i$ and $_e$ are the axial flows of neutral gas, ions, and electrons, respectively; B(*x*) is the profile of the radial magnetic field; A is the anode, E is the channel exit, B is the transition from the space-charge sheath to the quasineutral channel, and S (that can coincide with E) is the sonic point for ions. Three regions are usually distinguished in the quasineutral channel: the anode presheath BC, the diffusion zone CD, and the ionization zone DE.

which has resulted from a combination of basic considerations, detailed numerical solutions, and examination of experimental data. We will in the following sections offer partial analyses for several of these regions.

8. **The Presheath and the Diffusion Zone**

In the region B-C-D, ionization is very weak, because T_e has fallen to a low value as the electrons lose energy in the ionization layer.

Setting

$\nu_{ion} = 0$,

we can form the ratio

$v_i \times \text{Equation}\,(29)/\text{Equation}\,(31)$:

$$\frac{v_i dv_i}{kdT_e} = -\frac{5}{2}m_i$$

or

$$\frac{m_i v_i^2}{2} + \frac{5}{2}kT_e = \text{constant} \qquad (35)$$

At x=0 (point B), we use (34):

$$\frac{1}{2}\frac{5}{3}kT_e(0) + \frac{5}{2}kT_e(0) = \text{constant} \rightarrow \text{constant} = \frac{10}{3}kT_e(0)$$

or

$$T_e = \frac{4}{3}T_e(0) - \frac{m_i v_i^2}{5k} \qquad (36)$$

This can now be substituted into (29):

$$\left(\frac{4}{3}\frac{kT_e(0)}{v_i^2} - \frac{4}{5}m_i\right)dv_i = m_e \nu_e \frac{v_e}{v_i}dx = m_e \nu_e \frac{\Gamma_e}{\Gamma_i}dx$$

Since Γ_e and Γ_i are constant in this region (no ionization), we can integrate again (assuming $\nu_e \simeq$ constant as well):

$$-\frac{4}{3}\frac{kT_e(0)}{v_i} - \frac{4}{5}m_i v_i = C + m_e \nu_e \left(\frac{\Gamma_e}{\Gamma_i}\right)_0 x \qquad (37)$$

At x=0

$$v_i = v_i(0) = -\sqrt{\frac{5}{3}\frac{kT_{e_0}}{m_i}},$$

giving

$$C = -\frac{8}{5}m_i v_i(0).$$

Then (37) can be written as

$$\left(\frac{v_i}{v_i(0)}\right)^2 - 2\left[1 + \frac{5\Gamma_e m_e \nu_e x}{8\Gamma_i m_i |v_i(0)|}\right]\left(\frac{v_i}{v_i(0)}\right) + 1 = 0 \qquad (38)$$

Defining a "pre-sheath thickness"

$$\boxed{x_{ps} = \frac{8}{5}\left(\frac{\Gamma_i}{\Gamma_e}\right)_0 \frac{m_i\left|v_i(0)\right|}{m_e\nu_e}} \tag{39}$$

the solution to (38) is

$$\boxed{\frac{v_i}{v_i(0)} = 1+\frac{x}{x_{ps}} \pm \sqrt{\left(1+\frac{x}{x_{ps}}\right)^2 - 1}} \qquad \text{(select lower sign)} \tag{40}$$

$$= \frac{1}{1+\frac{x}{x_{ps}} \mp \sqrt{}}$$

Notice that for $x >> x_{ps}$, the upper sign choice leads to $\frac{v_i}{v_i(0)} \to \frac{2x}{x_{ps}}$ while the lower choice gives $\frac{v_i}{v_i(0)} \to \frac{x_{ps}}{2x}$. Since $v_i(0) < 0$, the first of these would imply a reverse ion flow which is <u>decelerating</u> towards sonic, and is therefore unphysical. For that reason, the lower sign has been selected. Since $n_e = \frac{(\Gamma_i)_0}{v_i}$, we also have then

$$\boxed{n_e = \frac{\left(\Gamma_i(0)/v_i(0)\right)}{1+\frac{x}{x_{ps}} - \sqrt{\left(1+\frac{x}{x_{ps}}\right)^2 - 1}}} = \frac{\Gamma_i(0)}{v_i(0)}\left[1+\frac{x}{x_{ps}} + \sqrt{\left(1+\frac{x}{x_{ps}}\right)^2 - 1}\right] \tag{41}$$

Returning to (36), we can now calculate also

$$\boxed{\frac{T_e}{T_e(0)} = 1+\frac{2}{3}\frac{\sqrt{\left(1+\frac{x}{x_{ps}}\right)^2 - 1}}{1+\frac{x}{x_{ps}} + \sqrt{\left(1+\frac{x}{x_{ps}}\right)^2 - 1}}} \tag{42}$$

The electric field follows from (32), for example. It is actually easier to return to (26) and integrate it (with $\nu_{ion} = 0$) to

$$\frac{m_i v_i^2}{2} + e\phi = \frac{m_i v_i^2(0)}{2} \tag{43}$$

and then, using (40),

$$\phi = \frac{5}{3}\frac{kT_e(0)}{e}\frac{\sqrt{\left(1+\frac{x}{x_{ps}}\right)^2 - 1}}{1+\frac{x}{x_{ps}}+\sqrt{\left(1+\frac{x}{x_{ps}}\right)^2 - 1}} \tag{44}$$

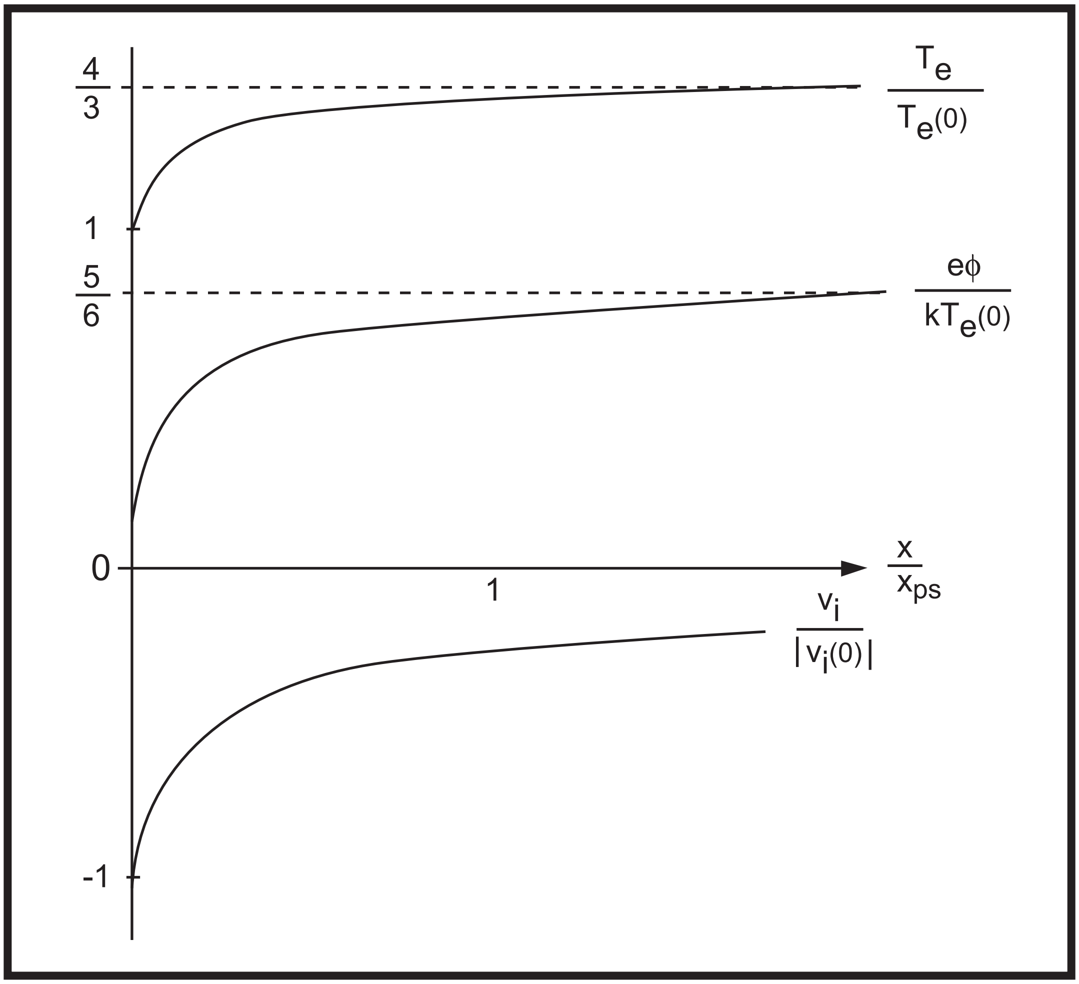
and, by differentiation,

$$E_x = \frac{-\frac{5}{3}\frac{kT_e(0)}{ex_{p_s}}}{\sqrt{\left(1+\frac{x}{x_{p_s}}\right)^2 - 1}\left[1+\frac{x}{x_p}+\sqrt{\left(1+\frac{x}{x_{p_s}}\right)^2 - 1}\right]^2} \tag{45}$$

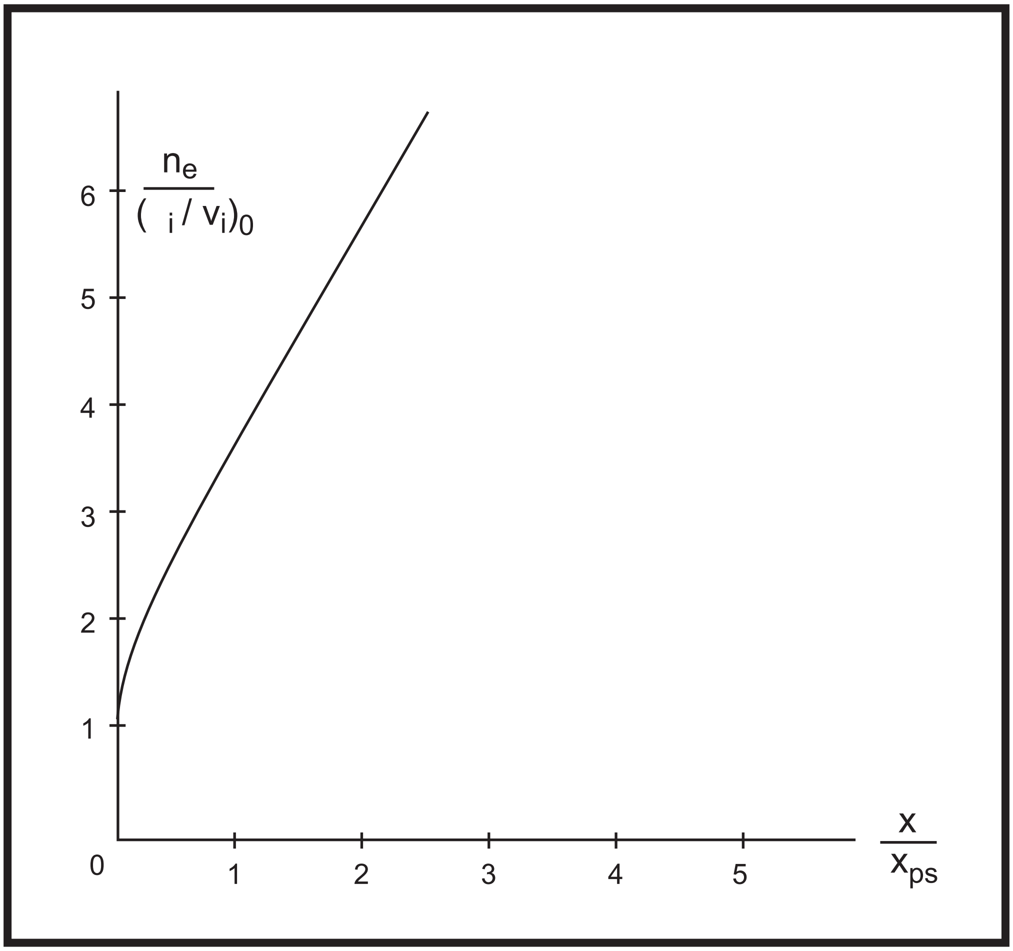
The limits of these expressions when $x >> x_{ps}$ (namely, in the diffusion region), are simple:

$$\frac{x}{x_{ps}} \rightarrow \infty; \quad \frac{v_i}{\left|v_i(0)\right|} \rightarrow -\frac{x_{ps}}{2x}; \quad n_e \rightarrow \frac{\Gamma_i(0)}{v_i(0)}\frac{2x}{x_{ps}}$$

$$\frac{T_e}{T_e(0)} \rightarrow \frac{4}{3}; \quad \phi \rightarrow \frac{5}{6}\frac{kT_e(0)}{e}; \quad E_x = -\frac{5}{12}\frac{kT_e(0)}{e}\frac{x_{p_s}^2}{x^3} \tag{46}$$



4/3
1
5/6
0
-1
1
Te / Te(0)
eϕ / kTe(0)
x / xps
vi / |vi(0)|



Notice how the electric field is very weak, except right near the anode, within the presheath, where the ions accelerate to sonic speed. Also, the temperature reaches a constant value outside the presheath, at 4/3 the sheath-edge value (this is $\frac{\gamma+1}{2}$ with $\gamma = \frac{5}{3}$). The electron density increases linearly in the diffusion region: ions need to advance towards the anode (in order to maintain neutrality), and since the field is too weak to move them against the collisional drag, a density gradient must appear so as to produce diffusion instead.

Combining the asymptotic form $n_e \simeq \frac{\Gamma_i(0)}{v_i(0)}\frac{2x}{x_{ps}}$ with Equation (39) for x_{ps}, we can write for the diffusion region

$$n_e \simeq \frac{3}{4}\frac{|\Gamma_e(0)|}{kT_e(0)} m_e \nu_e x \tag{47}$$

(NOTE: If B is non-uniform, so that $\nu_e = \nu_e(x)$, the factor $\nu_e x$ is simply replaced by $\int_0^x \nu_e dx$).

9. **The Transition to the Ionization Layer**

As x increases away from the anode, n_e cannot continue to increase indefinitely according to (47), because its maximum value must be of the order of $\Gamma_m / v_{i,exit}$, assuming near-full ionization. The end of the diffusion layer will be marked by an increase in T_e that will initiate ionization. Eventually, a forward field E_x will develop as well, but the analysis allowing both T_e and ϕ to vary as well as ionization to occur, is too complicated. We analyze first the portion of the ionization layer in which T_e rises, but E_x remains negligible. In this region, the form (24) of the energy equation is useful:

$$\Gamma_e\left(\frac{5}{2}kT_e + E_i'\right) = \text{constant}$$

Evaluate the constant in the diffusion region, where $T_e = \frac{4}{3}T_e(0)$ and $\Gamma_e = \Gamma_e(0)$:

$$\frac{\Gamma_e}{\Gamma_e(0)} = \frac{\frac{10}{3}kT_e(0) + E_i'}{\frac{5}{2}kT_e + E_i'} \tag{48}$$

In the diffusion layer, $kT_e << \frac{3}{10}E_i' \sim \frac{3}{4}E_i$, and so, even though (48) indicates a change in Γ_e as soon as T_e increases, the change will be slight at first. We can then approximately integrate (27), with ν_e = constant and $E_x = 0$, to

$$n_e kT_e \simeq -m_e \Gamma_e \nu_e x \qquad \text{(notice this is implied in Equation 47)} \tag{49}$$

From (48),

$$-\frac{\frac{5}{2}k\frac{dT_e}{dx}}{\frac{5}{2}kT_e + E_i'} = \frac{1}{\Gamma_e}\frac{d\Gamma_e}{dx} = \frac{n_e v_{ion}}{\Gamma_e}$$

and, using (49),

$$xdx \simeq \frac{k^2 T_e dT_e}{m_e \nu_e \nu_{ion}\left(kT_e + \frac{2}{5}E_i'\right)} \tag{50}$$

In the range where $kT_e << E_i$, the strong variability of ν_{ion} with T_e appears (see Equations 5-7) in the exponential e^{-E_i/kT_e}, if we write for now $\nu_{ion} \simeq ce^{-E_i/kT_e}$, then

$$\frac{k^2 T_e dT_e}{\nu_{ion}} \simeq \frac{k^2}{c} T_e e^{\frac{E_i}{kT_e}} dT_e \simeq -\frac{(kT_e)^3}{E_i} d\left(\frac{1}{\nu_{ion}}\right) \simeq -\frac{1}{E_i} d\left[\frac{(kT_e)^3}{\nu_{ion}}\right]$$

where the last step ignores a weaker term due to $\frac{1}{\nu_{ion}} d(kT_e)^3$. With these approximations, (50) integrates to

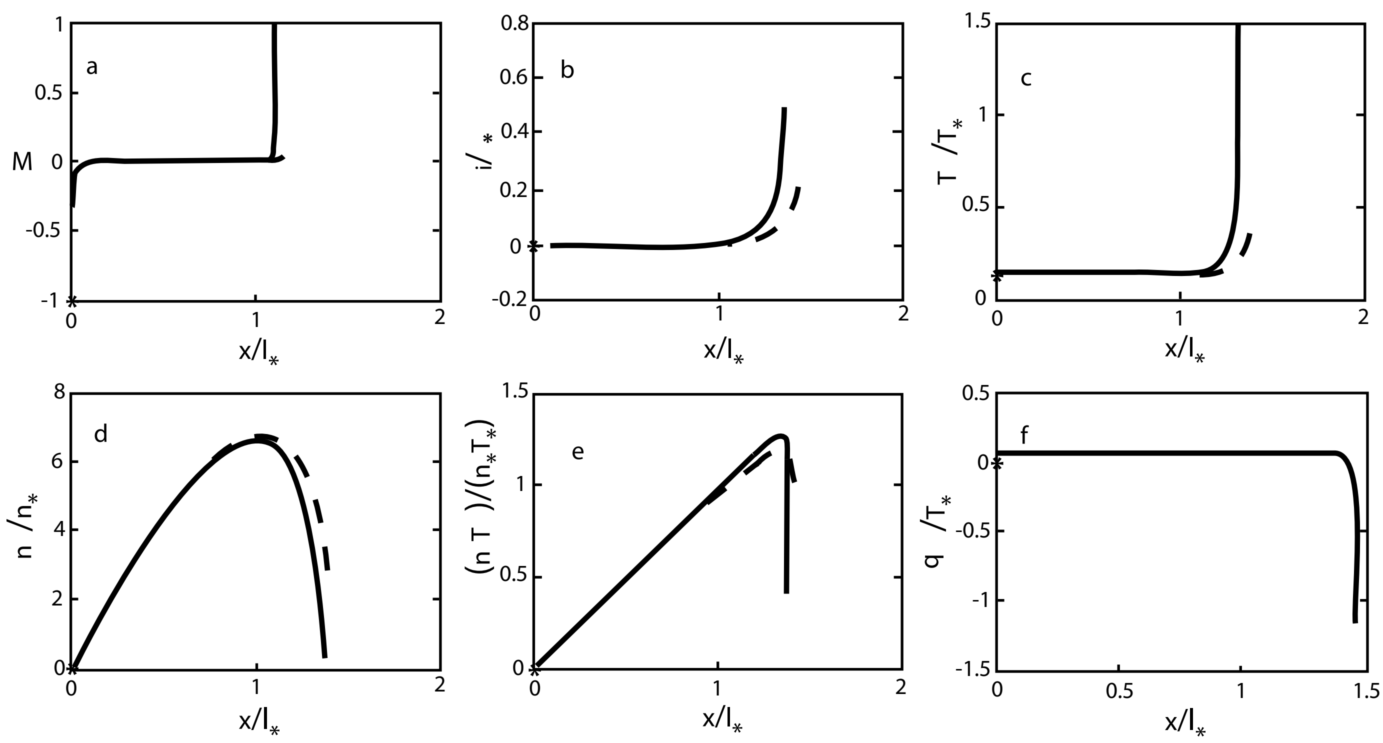
$$L^2 \simeq \left[\frac{2(kT_e)^3}{m_e \nu_e \nu_{ion} E_i \left(kT_e + \frac{2}{5} E_i'\right)}\right]_{T_e \simeq \frac{4}{3} T_e(0)} \qquad (51)$$

where L is the channel length (mainly its diffusion part), and T_e, K are evaluated in the diffusion region, which contributes the most to the integral. Given the length L, the B field, etc, Equation (51) determines $T_e(0)$, and therefore $v_i(0)$ and other quantities required in the previous section.

Figure (2) shows numerical results of integrating the full set of differential equations, as well as (dotted) those obtained using the approximations of the last two sections. Notice in particular that the rise of T_e, once started, is quite rapid, and leads to peak values $T_{e_{MAX}} \sim E_i$, at which ionization is rapid. The main body of the ionization layer is therefore very thin.

A quick argument for sonic exit to the ionization layer

In (29), <u>assume</u> (to be verified) that $\frac{5}{3} m_e \nu_e v_e v_i << \frac{5}{3} kT_e (\nu_{ion})_{MAX}$. Notice the bracket in (29) is >0 (since $v_e < 0$). Then, given the inequality above, the only way the RHS can cross zero if $\nu_{ion} << (\nu_{ion})_{MAX}$. The zero-crossing occurs at a smooth sonic point (where the LHS = 0). On the subsonic side of this, LHS>0, so this must be where the $\nu_{ion}[...] > 0$ term dominates; conversely, on the supersonic side, $\nu_{ion}[...]$ must become negligible. This means by the time the sonic point is reached, ν_{ion} <u>has fallen to nearly zero</u>, which can only be because n_n has been depleted - full ionization.



Plasma profiles for a channel with choked exit, no wall losses (l*/h = 0), uniform B-field, c / * = 1, d / m = 1, T_{eB} / E_i = 0.10, iB / m = -0.01, and * = m. Solid and dashed lines correspond to the exact and the approximate solution, respectively. Asterisks in figures correspond to point B.

References:

Ahedo, E., and M. Martinez-Sanchez. "AIAA 98-8788, One–dimensional Plasma Structure in Hall Thrusters." 34th AIAA/ASME/SAE/ASEE Joint Propulsion Conference and Exhibit. Cleveland, OH, July 13-15, 1998, E.T.S.I. Aeronáuticos Universidad Politécnica, Madrid, Spain and Massachusetts Institute of Technology, Cambridge, MA.