Lecture 16: Ion Engine Performance. Brophy's Theory

DEFINITIONS:

J_B=Beam ion (and neutralizer electron current)

J_E=cathode emitted current

J_c=ion current to cathode-potential surfaces

J_D=current through disch. power supply

J_P=total ion production rate

J_{ia}=Ion current to anode

J_{acc}=Ion current intercepted by accel. grid

Current Balances:

$$J_D = J_E + J_c + J_B + J_{acc}$$

and also $J_D = J_P + J_E - J_{ia}$

$$J_P = J_B + J_C + J_{ia} + J_{acc} \qquad \text{(ion balance)}$$

Useful Power = $J_B\left(V_B + V_D\right)$

Total Power = $J_B V_B + J_D V_D + J_{acc} V_B^{'} + P_{Heaters}$

Energy cost per beam ion

$$\varepsilon_B = \frac{\text{Total P. - Useful P.}}{J_B} = \frac{\left(J_D - J_B\right)V_D + J_{acc}V_B^{'} + P_H}{J_B}$$

$$P_H = J_E V_c$$

$$J_D - J_B = J_E + J_c + J_{acc}$$

$$\varepsilon_B = \left(\frac{J_E V_D + P_H}{J_P}\right)\frac{J_P}{J_B} + \frac{J_c V_D + J_{acc}\left(V_B^{'} + V_D\right)}{J_B}$$

$$\varepsilon_p$$

$$\boxed{\varepsilon_B = \frac{\varepsilon_p}{f_B} + \frac{f_c}{f_B}V_D + \frac{f_{acc}}{f_B}\left(V_B^{'} + V_D\right)} \qquad \text{(plasma ion cost)}$$

where $f_B = \frac{J_B}{J_p}, f_c = \frac{J_c}{J_p}, f_{acc} = \frac{J_{acc}}{J_p}$ and $\varepsilon_p = \frac{J_E V_D + P_H}{J_p}$

More definitions:

U_+ = ionization energy per ion

U_j = excitation energy of level j

J_j = excitation rate (total)

J_{LP} = loss rate of primary electrons

ε_m = mean energy of Maxwellian electron group

Discharge Energy Balance

$$U_+ + \sum_j J_j \frac{U_j}{J_P} + \frac{J_{Lp}V_D}{J_P} + \varepsilon_m \frac{\left(J_p + J_E - J_{ia} - J_{Lp}\right)}{J_P} = \frac{J_E V_D}{J_p} = \varepsilon_p - \frac{P_H}{J_p}$$

Define $\varepsilon_o = U_+ + \sum_j J_j \frac{U_j}{J_p}$. Then

$$-\frac{P_H}{J_P} + \varepsilon_p = \varepsilon_o + \varepsilon_m + \left(V_D - \varepsilon_m\right)\frac{J_{Lp}}{J_E}\frac{J_E}{J_p} - \frac{J_{ia}}{J_p}\varepsilon_m + \varepsilon_m\left(\frac{\varepsilon_p - P_H / J_p}{V_D}\right)$$

Use $\frac{\varepsilon_p - P_H / J_P}{V_D}$

$$\varepsilon_p\left[1 - \frac{V_D - \varepsilon_m}{V_D}\frac{J_{Lp}}{J_E} - \frac{\varepsilon_m}{V_D}\right] = \varepsilon_o + \varepsilon_m - \left(V_D - \varepsilon_m\right)\frac{J_{Lp}}{J_E}\frac{P_H}{J_p V_D} - \frac{J_{ia}}{J_p}\varepsilon_m - \varepsilon_m\frac{P_H}{J_p V_D} + \frac{P_H}{J_p V_D} + \frac{P_H}{J_p}$$

$$1 - \frac{J_{Lp}}{J_E} - \frac{\varepsilon_m}{V_D}\left(1 - \frac{J_{Lp}}{J_E}\right)$$

$$1 - \left(1 - \frac{J_{Lp}}{J_E}\right)\left(1 - \frac{\varepsilon_m}{V_D}\right)$$

$$\varepsilon_p\left(1-\frac{J_{Lp}}{J_E}\right)\left(1-\frac{\varepsilon_m}{V_D}\right)=\varepsilon_o+\varepsilon_m-\frac{P_H}{J_p}\left(\frac{V_D-\varepsilon_m}{V_D}\frac{J_{Lp}}{J_E}+\frac{\varepsilon_m}{V_D}-1\right)-\frac{J_{ia}}{J_p}\varepsilon_m$$

$$+\left(1-\frac{J_{Lp}}{J_\varepsilon}\right)\left(1-\frac{\varepsilon_m}{V_p}\right)\frac{P_H}{J_p}$$

$$\boxed{\varepsilon_p=\frac{\varepsilon_o+\varepsilon_m\left(1-\frac{J_{ia}}{J_p}\right)}{\left(1-\frac{J_{Lp}}{J_E}\right)\left(1-\frac{\varepsilon_m}{V_D}\right)}+\frac{P_H}{J_p}}$$

NOTE: If we write $P_H = J_E V_c$, the equation for ε_p becomes

$$\varepsilon_p=\frac{\varepsilon_o+\varepsilon_m\left(1-\frac{J_{ia}}{J_p}\right)}{\left(1-\frac{J_{Lp}}{J_E}\right)\left(1-\frac{\varepsilon_m}{V_D}\right)}\left(1+\frac{V_C}{V_D}\right)$$

Survival Equation for Primary Electrons

$\frac{J_{Lp}}{J_E}=e^{-\sigma_{tot}n_n\ell_e}$, where $\sigma_{tot}=\left(\sigma_+ + \sigma_{exc}\right)_{primaries}$. Also, ℓe is the path length for a

primary electon before it would be captured by the anode, if it did not collide with a neutral before before that. This path lenght is that of the electron's helical path around one of the magnetic lines of force created by the confinement magnets.

The neutral density is related to the flow rate by

$$n_n=\frac{\Gamma_n}{\frac{c_n}{4}\phi}=\frac{\dot{m}}{m_i A_g}\left(1-\eta_u\right)\frac{4}{c_n\phi}$$

(ϕ = grid system transparency for neutrals; η_u = utilization efficiency)

$$\frac{J_{LP}}{J_E}=\exp\left[-\frac{4\dot{m}\left(1-\eta_u\right)\sigma\,\ell_e}{m_i A_g c_n\phi}\right]$$

$$\varepsilon_p = \frac{\varepsilon_p^*}{1 - \exp\left[-c_o \dot{m}(1-\eta_n)\right]}$$

where $$\varepsilon_p^* = \frac{\left[\varepsilon_o + \varepsilon_m(1-f_{ia})\right]\left(1 + \frac{V_c}{V_D}\right)}{1 - \frac{\varepsilon_m}{V_D}}$$, and $$C_o = \frac{4\,\sigma_T \ell e}{m_i A_g c_n \phi}$$

The quantity C_o is a measure of the confinement effectivenss for primary electrons (better for long electron path ℓe, small grid open area $A_g\phi$). If $C_o \to \infty$, the energy cost per ion, ε_p, tends to the limit ε_p^*, which then represents the cost per ion no primary losses.

Calculation of Primary / Secondary Population Ratio

Primaries are endowed initially with an energy V_D, and, if they did not escape, would all thermalize eventually, to an energy ε_m. The rate at which they disappear in that case is simply the rate of ionization or excitation by primaries (a primary is assumed to become a secondary-Maxwellian - after one ionization or one excitation). So, the net energy input rate per unit volume due to injection of primaries is (without escape)

$$n_n n_p \upsilon_p \sigma_T(V_D)(V_D - \varepsilon_m) \qquad (\sigma_T = \sigma_+ + \sigma_{exc})$$

This energy is used by the primaries and their secondary “progenie” to

(a) Produce ionization by primaries. Per ionization event, this uses
$U^+ + E_m$, since the new electron created has energy ε_m.

Total p.u. volume $\quad n_n n_p \upsilon_p \sigma_T(V_D)(U^+ + \varepsilon_m)$

(b) Excite atom, by primaries Total energy rate p.u. volume $n_n n_p \upsilon_p \sigma_{exc}(v_o)U_{exc}$

(a shorthand for $n_p \upsilon_p \sum_j \sigma_j U_j$)

(c) Produce ionization by secondaries (Mawellian). The rate p.u. volume is

$$n_n \int_o^\infty f_m(c) c \sigma_+(c) 4\pi c^2 dc \equiv n_n \bar{c}_e n_m \bar{\sigma}_+$$

where $$\bar{\sigma}_+ = \frac{1}{n_m \bar{c}_e} \int_o^\infty f_m(c) c\, \sigma_+(c) 4\pi c^2 dc$$

and $$\bar{c}_e = \sqrt{\frac{8}{\pi} \frac{kTe}{m_e}} \qquad , \qquad n_m = \int_o^\infty f_m\, 4\pi c^2 dc$$

The ionization cross-section $\sigma_t(c)$ in zero below $c^+ = \sqrt{\frac{2eU^+}{m_e}}$. Using a Maxwellian form for $f_m(c)$, we find easily

$$\bar{\sigma}_t = \int_{u^+}^\infty e^{-u} u\, \sigma_+(u) du \qquad \left(u = \frac{E}{kT_e}\right)$$

and the energy spent by secondaries in ionization (p.u. time and volume) is then

$$n_n n_m \bar{c}_e \bar{\sigma}_+ \left(U^+ + \varepsilon_m\right)$$

Similarly, the energy spent in excitation is

$$n_n n_m \bar{c}_e \bar{\sigma}_{exc} U_{exc}$$

The energy balance is therefore (dividing by n_n throughout)

$$n_p \upsilon_p \sigma_+ \left(V_D - \varepsilon_m\right) = n_p \upsilon_p \left[\sigma_+(V_D)\left(U^+ + \varepsilon_m\right) + \sigma_{exc}(V_D) U_{exc}\right] + n_m \bar{c}_e \left[\bar{\sigma}_+ \left(U^+ + \varepsilon_m\right) + \bar{\sigma}_{exc} U_{exc}\right]$$

This can be solved for $\frac{n_p}{n_m}$:

$$\boxed{\frac{n_p}{n_m} = \frac{\bar{c}_e}{\upsilon_p} \frac{U^+ + \varepsilon_m + U_{exc} \frac{\bar{\sigma}_{exc}}{\bar{\sigma}_+}}{\left(V_D - \varepsilon_m\right) \frac{\sigma_T(V_D)}{\bar{\sigma}_+} - \frac{\sigma_+(V_D)}{\bar{\sigma}_+}\left(U^+ + \varepsilon_m\right) - \frac{\sigma_{exc}(V_D)}{\bar{\sigma}_+}}}$$

which is a function of T_e for a fixed V_D. Hence ε_p^* is also a function of T_e. This is because, given

$$\varepsilon_p = \frac{\varepsilon_p^*}{1 - \exp\left[-C_o \dot{m}\left(1 - \eta_u\right)\right]}$$

ε_p^* is seen to be the energy per ion created if no primary electrons were to escape $(\varepsilon_o \to \infty)$. The expression for ε_p^* (neglecting ion capture by the screen, $f_{ia} = o$, and heating power, $V_c = o$) was

$$\varepsilon_p^* = \frac{\varepsilon_o + \varepsilon_m}{1 - \frac{\varepsilon_m}{V_D}} \quad ; \quad \varepsilon_o = U_+ + \frac{J_{exc}}{J_p} U_{exc}$$

or

$$\varepsilon_o = U^+ + U_{exc} \frac{n_p \upsilon_p \sigma_{exc}(V_D) + n_m \bar{c}_e \bar{\sigma}_{exc}}{n_p \upsilon_p \sigma_+(V_D) + n_m \bar{c}_e \bar{\sigma}_+}$$

and this does depend on T_e and $\frac{n_p}{n_m}(T_e)$. Substituting the expression for $\frac{n_p}{n_m}$ found above, and simplifying, we obtain

$$\boxed{\varepsilon_p^* = V_D \sigma_T(V_D) \frac{\bar{\sigma}_+ (U^+ + \varepsilon_m) + \bar{\sigma}_{exc} U_{exc}}{[\bar{\sigma}_{exc} \sigma_+(V_B) - \sigma_{exc}(V_D) \bar{\sigma}_+] U_{exc} + \bar{\sigma}_+ \sigma_T(V_D)(V_D - \varepsilon_m)}}$$

NOTE: An intermediate expression for ε_p^* (still containing $\frac{n_p}{n_m}$), which will be useful later, is

$$\varepsilon_p^* = \frac{V_D \left(\frac{\sigma_T}{\sigma_+}\right)}{1 + \frac{n_m}{n_p} \frac{\bar{c}_e}{\upsilon_p} \frac{\bar{\sigma}_+}{\sigma_+(V_D)}}$$

<u>Calculation of utilization efficiency</u>

$$\frac{J_p}{e} = (Vd)(n_p \upsilon_p \sigma_+(V_D) + n_m \bar{c}_e \bar{\sigma}_+) n_n \qquad J_B = f_B J_p = \frac{e}{m_i} \eta_u \dot{m}$$

$$\uparrow \dot{m}(1 - \eta_u) \frac{4}{\bar{c}_n \phi A_g m_i}$$

$$V(n_p \upsilon_p \sigma_+(V_D) + n_m \bar{c}_e \bar{\sigma}_+) \frac{4(1 - \eta_u)}{\bar{c}_n \phi Ag} = \frac{\eta_u}{f_B}$$

But also $n_+ = n_m + n_p$ and $J_B = e\, n_+ 0.61 \upsilon_B A_g \phi_i$

so $n_m + n_p = \dfrac{J_B}{e\upsilon_B Ag\phi_i}$

Divide: $$V\,\bar{c}_e\bar{\sigma}_+ \frac{1+\dfrac{n_p}{n_m}\dfrac{\upsilon_p(V_D)}{\bar{c}_e}\dfrac{\sigma_+(V_D)}{\bar{\sigma}_+}}{1+\dfrac{n_p}{n_m}}\frac{4(1-\eta_n)}{\bar{c}_n\phi Ag} = \frac{e\,0.6A_g\phi_i}{J_B f_B}\eta_u$$

$$\frac{1-\eta_u}{\eta_u} = \frac{e\,0.61\upsilon_B A_g^2\phi\phi_i\bar{c}_n\left(1+\dfrac{n_p}{n_m}\right)}{4J_B f_B V\bar{c}_e\bar{\sigma}_+\left(1+\dfrac{n_p}{n_m}\dfrac{\upsilon_p}{\bar{c}_e}\dfrac{\sigma_+(V_D)}{\bar{\sigma}_+}\right)} = y \quad \rightarrow \quad \boxed{\eta_u = \frac{1}{1+y}}$$

$$\frac{V_D}{\varepsilon_P^*}\frac{\sigma_T(V_D)}{\bar{\sigma}_+}\frac{n_p}{n_m}\frac{\upsilon_p}{\bar{c}_e}$$

$$\boxed{y = \frac{0.15\,e\,\varepsilon_p^*\upsilon_B A_g^2\phi\phi_i\bar{c}_n\left(1+\dfrac{n_p}{n_m}\right)}{J_B f_B V_D\upsilon_p V\sigma_T(V_D)\left(\dfrac{n_p}{n_m}\right)}}$$

<u>So:</u> Given $V_D, J_B, T_{walls} \rightarrow \bar{c}_n$

Geometry: f_B, ϕ, ϕ_i, V, Ag

Gas $\left(U^+, U_{exc}, \sigma_+(E), \sigma_{exc}(E), M\right)$

Can conclude $\dfrac{n_p}{n_m}(T_e) \rightarrow \varepsilon_p^*(T_e) \rightarrow \eta_u(T_e)$ NOTE: $\dot{m} = \dfrac{m_i}{e}\dfrac{J_B}{\eta_n}$ so real parameter is $\dot{m}$, not J_B.

Then given <u>also</u> ℓ_e (magnetic geometry) and $\dot{m} \rightarrow C_o \rightarrow \varepsilon_p \rightarrow \varepsilon_B$

<u>Summary</u>

Inputs:

Geometry and Magentic Field	Gas Properties	Operating Parameters	Additional Parameter
$f_B, f_c, \ldots$ ϕ, ϕ_i, V, Ag ℓe	U_+, U_{exc} $\sigma_+(E), \sigma_{exc}(E)$ M	V_D, J_B $T_w \rightarrow \bar{c}_n$	$T_e \rightarrow \varepsilon_m$

$$\frac{n_p}{n_m} = \frac{\bar{c}_e}{\upsilon_p} \frac{U_+ + \varepsilon_m + U_{exc} \frac{\sigma_{exc}}{\bar{\sigma}}}{(V_D - \varepsilon_m) \frac{\sigma_+(V_D)}{\bar{\sigma}_+}} U_{exc}$$

↓

$$\varepsilon_T^* = V_D \left(\frac{\sigma_T}{\sigma_+} \right)_{V_D} \frac{\frac{n_e}{n_m} \frac{\upsilon_p}{\bar{c}_e} \frac{\sigma_+(V_D)}{\bar{\sigma}_+}}{1 + \frac{n_p}{n_m} \frac{\upsilon_p}{\bar{c}_e} \frac{\sigma_+(V_D)}{\bar{\sigma}_+}}$$

↓

$$Y = \frac{0.15\, e\, \varepsilon_p^* \upsilon_B A_g^2 \phi \phi_i \bar{c}_n \left(1 + \frac{n_e}{n_m}\right)}{J_B f_B V_D \upsilon_p V \sigma_T (V_D) \left(\frac{n_p}{n_m}\right)} \quad ; \quad \eta_u = \frac{1}{1 + Y} \quad \rightarrow \quad \dot{m} = \frac{m_i}{e} \frac{J_B}{\eta_u}$$

$$C_o = \frac{4 \sigma_T (V_D) \ell e}{m_i A_g \bar{c}_n \phi} \quad \rightarrow \quad \varepsilon_p = \frac{\varepsilon_p^*}{1 - \exp[-C_o \dot{m} (1 - \eta_u)]}$$

↓

$$\varepsilon_B = \frac{\varepsilon_p}{f_B} + \frac{f_c}{f_B} V_D$$