16.522, Space Propulsion
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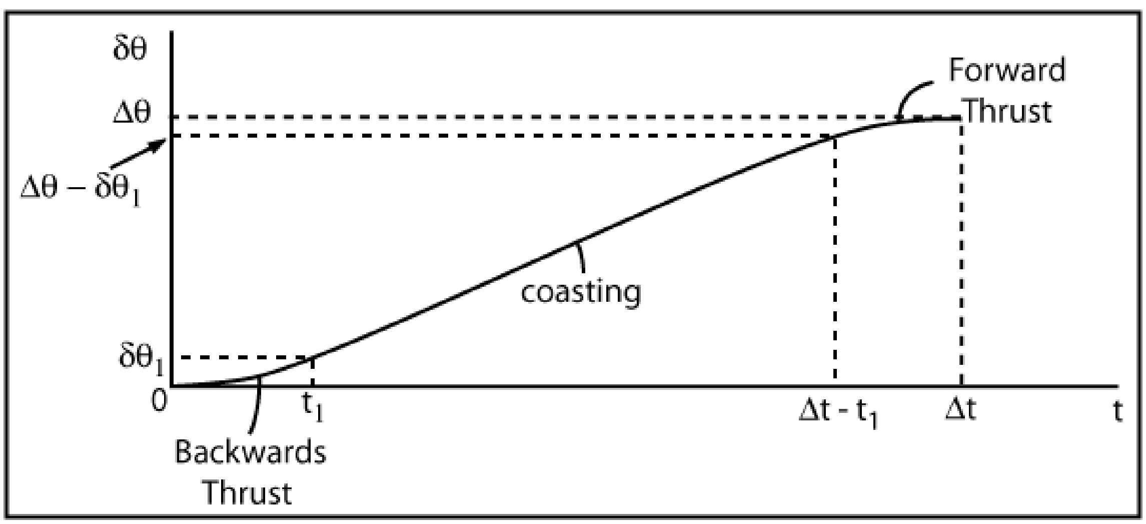
Lecture 4: Re-positioning in Orbits

Suppose we want to move a satellite in a circular orbit to a position $\Delta\vartheta$ apart in the same orbit, in a time Δt (assumed to be several orbital times at least). The general approach is to transfer to a lower (for positive $\Delta\vartheta$) or higher (for $\Delta\vartheta < 0$) nearby orbit, then drift in this faster (or slower) orbit for a certain time, then return to the original orbit.

The analysis is similar for low and high thrust, because we have radius ratios very close to 1, so that, in either case the satellite is nearly "in orbit" even during thrusting periods, and $\Delta V's$ for orbit transfer amount (in magnitude) to the difference of the beginning and ending orbital speeds. In detail, of course, if done at high thrust the maneuver involves a two-impulse Hohmann transfer to the drift orbit and one other two-impulse Hohmann transfer back to the original orbit. For the low-thrust case, continuous thrusting is used during both legs, with some guidance required to remove the very slight radial component of $\vec{v}$ picked up during spiral flight (although ignored here).

We will do the analysis for the low-thrust case only, then adapt the result for high-thrust.

Let $\delta\vartheta$ be the advance angle relative to a hypothetical satellite remaining in the original orbit and left undisturbed. The general shape of the maneuver is sketched below:



Since the orbital angular velocity is

$$\Omega = \sqrt{\frac{\mu}{r^3}},$$

its variation with orbit radius is

$$\delta\Omega = -\frac{3}{2}\Omega\frac{\delta r}{r} = \frac{d\left(\delta\vartheta\right)}{dt} \tag{1}$$

During thrusting, δr is varying according to

$$\frac{d}{dt}\left(-\frac{\mu}{2r}\right) \cong \frac{F}{M}v \cong a\sqrt{\frac{\mu}{r}} \qquad \left(a = \frac{F}{M}\right) \tag{2}$$

or

$$\frac{\mu}{2r^2}\frac{dr}{dt} \cong a\sqrt{\frac{\mu}{r}} \rightarrow \frac{1dr}{rdt} = \frac{2ar^{1/2}}{\sqrt{\mu}} \tag{3}$$

and so

$$\frac{d\left(\delta\Omega\right)}{dt} = \frac{d^2\left(\delta\vartheta\right)}{dt^2} = -\frac{3}{2}\Omega\frac{2ar^{1/2}}{\sqrt{\mu}} = -\frac{3a}{r} \tag{4}$$

For integration, we will regard $r \cong r_0$ as a constant (small variations):

$$\frac{d\left(\delta\vartheta\right)}{dt} = -\frac{3a}{r_0}t + constant \tag{5}$$

Starting from t $= 0$, $\delta\vartheta = 0$, $\frac{d\left(\delta\vartheta\right)}{dt} = 0$,
we obtain

$$\frac{d\left(\delta\vartheta\right)}{dt} = -\frac{3a}{r_0}t \ ; \ \delta\vartheta = -\frac{3}{2}\frac{a}{r_0}t^2 \left(t < t_1\right) \tag{6}$$

After (t=t_1), we continue to drift at a constant rate

$$\frac{d\left(\delta\vartheta\right)}{dt} = -\frac{3a}{r_0}t_1$$

and, since we start from

$$\delta\vartheta\left(t_1\right) = -\frac{3}{2}\frac{a}{r_0}t_1^2,$$

the $\delta\vartheta$ during the coasting phase is

$$\delta\vartheta = -\frac{3}{2}\frac{a}{r_0}t_1^2 - \frac{3a}{r_0}t_1\left(t - t_1\right) = -\frac{3at_1}{r_0}\left(t - \frac{t_1}{2}\right) \tag{7}$$

At the end of coasting $\left(t = \Delta t - t_1\right)$, we have then

$$\delta\vartheta(\Delta t\text{-}t_1) = -\frac{3at_1}{r_0}\left(\Delta t - \frac{3}{2}t_1\right) \qquad (8)$$

and, after the period t_1 of reversed thrust, we return to the initial orbit with $\frac{d(\delta\vartheta)}{dt} = 0$, and with $\delta\vartheta$ as in (8), plus a further $\delta\vartheta(t_1)$.

This is then the total $\Delta\vartheta$:

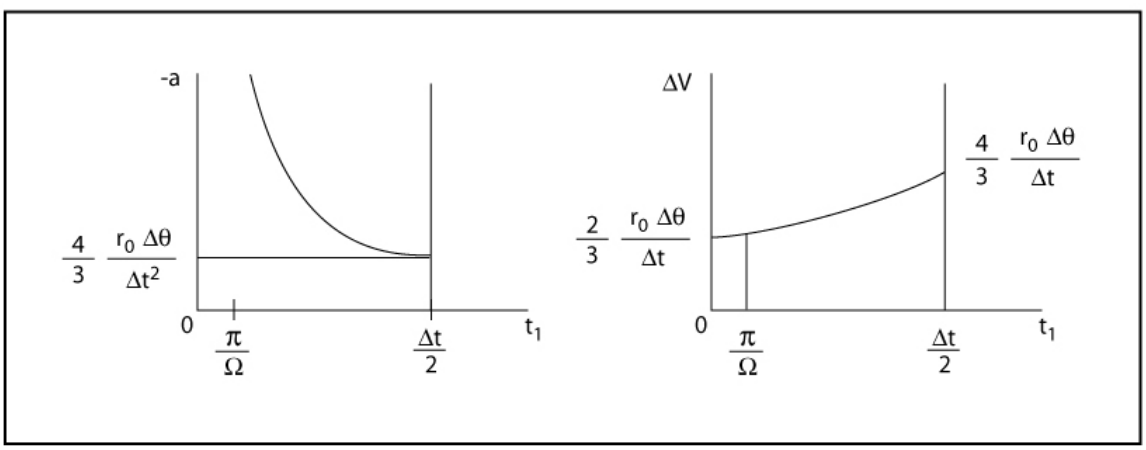
$$\Delta\vartheta = -\frac{3at_1}{r_0}\left(\Delta t - \frac{3}{2}t_1\right) - \frac{3}{2}\frac{a}{r_0}t_1^2$$

$$\boxed{\Delta\vartheta = -\frac{3at_1}{r_0}(\Delta t - t_1)} \qquad (9)$$

Clearly, the mission (given $\Delta\vartheta$ and Δt) can be accomplished with different choices of thrusting time t_1 (but notice that $t_1 < \Delta t/2$ in any case). The required thrust/mass ratio (a) and $\Delta V = 2|a|t_1$ depend on this choice:

$$\boxed{-a = \frac{1}{3}\frac{r_0\Delta\vartheta}{t_1(\Delta t - t_1)}} \qquad (10)$$

$$\boxed{\Delta V = \frac{2}{3}\frac{r_0\Delta\vartheta}{(\Delta t - t_1)}} \qquad (11)$$



We can see here that low thrust ends up as a penalty on ΔV, so that the <u>thrusting time should be selected as short as possible</u> within the available on-board power.

In the limit of impulsive thrust, we realize that t_1 cannot really be any less than the Hohmann transfer time (1/2 orbit, or $t_1 = \frac{\pi}{\Omega}$). A more detailed analysis of this case confirms that, for the high thrust case, Equations (10) and (11) are indeed valid with $t_1 = \pi/\Omega$.

The power per unit mass required is:

$$\frac{P}{M} = \frac{1}{2\eta}\frac{|F|c}{M} = \frac{|a|c}{2\eta}$$

$$\frac{P}{M} = \frac{1}{6\eta}\frac{r_0 \Delta\theta\, c}{t_1\left(\Delta t - t_1\right)}$$

It is useful to express results in terms of the coasting time,

$$t_c = \Delta t - 2t_1$$

$$t_1 = \frac{\Delta t - t_c}{2} \quad \Rightarrow \quad \Delta t - t_1 = \frac{\Delta t + t_c}{2},$$

$$t_1\left(\Delta t - t_1\right) = \frac{\Delta t^2 + t_c^2}{4},$$

We then have

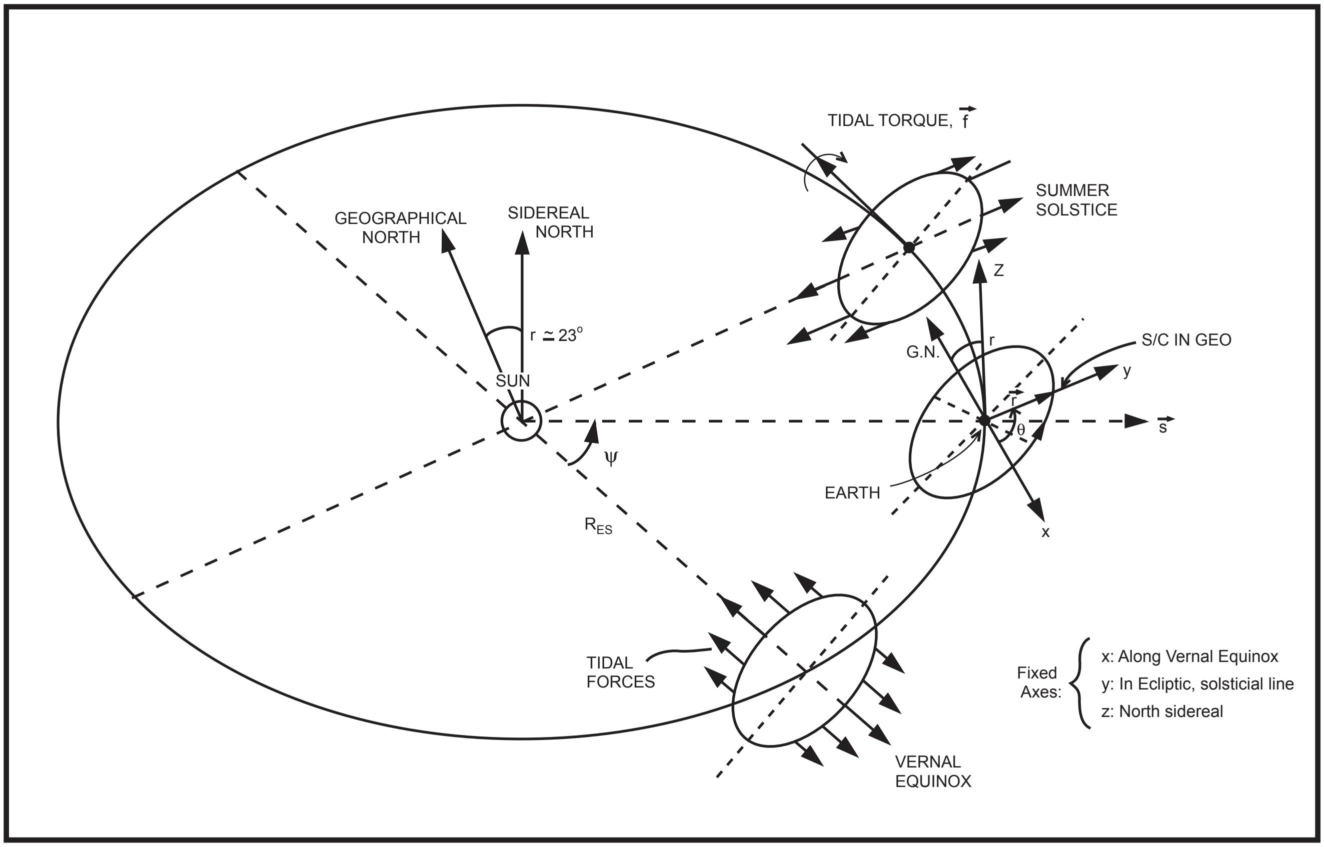
$$\Delta V = \frac{4}{3}\frac{r_0 \Delta\theta}{\Delta t + t_c}$$

$$\frac{P}{M} = \frac{2c}{3\eta}\frac{r_0 \Delta\theta}{\Delta t^2 - t_c^2}$$

coasting reduces ΔV, but increases P/M (not much if $t_c/\Delta t$ is small).

Sun-Moon Effects on a Geosynchronous Orbit – The N/S Drift

Geosynchronous orbits are in the Equatorial plane, which is inclined 23.44° to the plane of the Ecliptic (path of Earth around the Sun). Because of this, the tidal or gravity-gradient forces exerted by the Sun on a geosynchronous spacecraft will sometimes exert a "torque" on the orbit. This happens primarily at Solstice (January and July) when the orbit dips the most out of the Ecliptic:



If $\vec{r}$ is the relative position vector of the spacecraft, and $\vec{s}$ is the unit vector from the Sun, the Tidal Force (Gravity gradient) per unit mass is

$$\vec{f} = \frac{\mu_s}{R_{Es}^3}\left[3\left(\vec{r}.\vec{s}\right)\vec{s} - \vec{r}\right] \tag{1}$$

(This the imbalance between solar attraction and centrifugal force, which cancels on the Earth, $\vec{r} = 0$). Of this, the $\vec{r}$ part gives no torque about the Earth's center. To compute the effect on the orbit, we "smear" the torque $\vec{r} \times \vec{f}$ about the geosynchronous orbit, and calculate an orbit-average torque (per unit mass):

$$\vec{q} = \frac{1}{2\pi}\int_0^{2\pi} \vec{r} \times \vec{f}\, d\theta \tag{2}$$

$$\vec{q} = \frac{3\mu_s}{R_{Es}^3}\frac{1}{2\pi}\int_0^{2\pi} \left(\vec{r}.\vec{s}\right)\left(\vec{r} \times \vec{s}\right) d\theta \tag{3}$$

We project onto a set of axis (x, y, z) as defined in the sketch (fixed axis):

$$\vec{r} = \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = R_G \begin{Bmatrix} \cos\theta \\ \sin\theta\cos\gamma \\ \sin\theta\sin\gamma \end{Bmatrix} \tag{4}$$

where R_G is the geosynchronous orbit radius (42, 200 Km) and $\gamma = 23.44^\circ$ is the angle between the equatorial plane and the ecliptic.

Also

$$\vec{s} = \begin{Bmatrix} \cos\psi \\ \sin\psi \\ 0 \end{Bmatrix} \quad (5)$$

So that

$$\vec{r}.\vec{s} = R_G\left(\cos\psi\cos\theta + \sin\psi\cos\gamma\sin\theta\right) \quad (6)$$

$$\vec{r}\times\vec{s} = R_G \begin{Bmatrix} -\sin\psi\sin\gamma\sin\theta \\ \cos\psi\sin\gamma\sin\theta \\ \sin\psi\cos\theta - \cos\psi\cos\gamma\sin\theta \end{Bmatrix} \quad (7)$$

In performing the averaging of Equation (3), we keep ψ (approximately) constant. The Earth does not advance much in its orbit in one day. Also, of course, γ is constant. When (6) and (7) are multiplied together we obtain $\sin\theta$ and $\cos\theta$ quadratic combinations, and we use

$$\frac{1}{2\pi}\int_0^{2\pi}\sin\theta\cos\theta\, d\theta = 0 \;\; ; \;\; \frac{1}{2\pi}\int_0^{2\pi}\sin^2\theta\, d\theta = \frac{1}{2\pi}\int_0^{2\pi}\cos^2\theta\, d\theta = \frac{1}{2} \quad (8)$$

and therefore the mean torque is

$$\vec{q} = \frac{3}{2}\frac{\mu_S}{R_{ES}^3}R_G^2\sin\gamma\sin\psi \begin{Bmatrix} -\cos\gamma\sin\psi \\ \cos\gamma\cos\psi \\ \sin\gamma\cos\psi \end{Bmatrix} \quad (9)$$

This torque will precess the geosynchronous orbit by changing its specific angular momentum $\vec{\ell} = R_G^2\,\omega_G\,\vec{n}_G$ ($\vec{n}_G$ =unit vector along the Geographical North direction, $\omega_G = \frac{2\pi}{86400}$ rad/s).

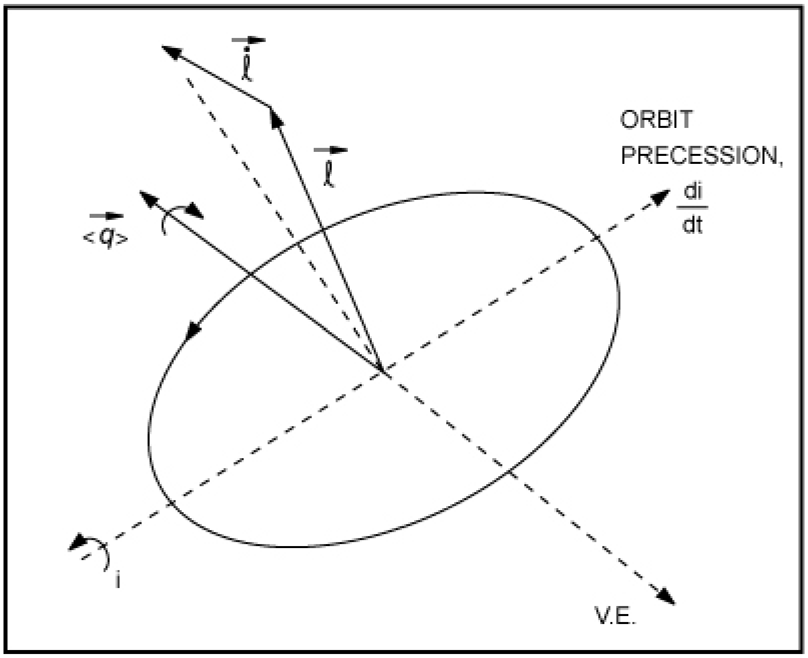
We are interested in the long-term (secular) effects. For times much longer than $\frac{1}{2}$ year, we can also average the torque over the angle ψ . Noticing that, as before,

$\left\langle \sin\psi\cos\psi \right\rangle = 0$,

$\left\langle \sin^2\psi \right\rangle = \frac{1}{2}$,

$$\left\langle \vec{q} \right\rangle_{secular} = -\frac{3}{4}\frac{\mu_s}{R_{ES}^3}R_G^2\sin\gamma\cos\gamma \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} \quad (10)$$

This shows a non-cancelling effect consisting on a mean torque which acts along the (negative) vernal equinox direction (as shown in the sketch):



The effect of this torque along (-x) is to add a small $\frac{d\vec{\ell}}{dt}$ to $\vec{\ell}$, along (-x), and hence to rotate (precess) the orbit along the axis perpendicular to the vernal direction, as shown.

In magnitude

$$\left|\frac{d\vec{\ell}}{dt}\right| = \left|\vec{\ell}\right|\frac{di}{dt} = R_G^2\omega_G\frac{di}{dt} \tag{11}$$

and equating to $\left|\left\langle\vec{q}\right\rangle\right|$ from (10),

$$\boxed{\left(\frac{di}{dt}\right)_{sun} = -\frac{3}{4}\frac{\mu_s}{R_{ES}^3}\frac{\sin\gamma\cos\gamma}{\omega_G}} \tag{12}$$

Or, since the angular frequency of the Earth's heliocentric motion is

$$\omega_{ES} = \sqrt{\frac{\mu_s}{R_{ES}^3}},$$

$$\boxed{\left(\frac{di}{dt}\right)_{sun} = -\frac{3}{4}\frac{\omega_{ES}^2}{\omega_G}\sin\gamma\cos\gamma} \tag{13}$$

Numerically, expressing ω_{ES} and ω_G in (deg/yr),

$$\left[\left(\frac{di}{dt}\right)_{sun} = -\frac{3}{4}\frac{(360)^2}{(360\times 365)}\sin 23.44^\circ \cos 23.44^\circ = -0.27^\circ/yr\right]$$

The Moon effect is similar, except that the Moon's orbital axis which is inclined at 5.15° with respect to sidereal North, precesses slowly about that direction (once every 18.6 yr.). This means the angle between the Equator and the Moon's orbit varies between $\gamma_M = 23.44^\circ + 5.15^\circ = 28.59^\circ$ and $\gamma_M = 23.44^\circ - 5.15^\circ = 18.29^\circ$.

The expression for $\left(\frac{di}{dt}\right)_{moon}$ is similar to (12);

$$\left(\frac{di}{dt}\right)_{moon} = -\frac{3}{4}\frac{\mu_M}{R_{EM}^3}\sin\gamma_m \cos\gamma_m \tag{14}$$

and

$$\frac{\mu_M}{R_{EM}^3} = \frac{\mu_M}{\mu_E}\frac{\mu_E}{R_{EM}^3} = \left(\frac{M_m}{M_E}\right)\omega_{ME}^2 \tag{15}$$

where $\frac{M_m}{M_E} = \frac{1}{81.3}$ is the Moon/Earth mass ratio, and ω_{ME} is the angular velocity of the Moon about Earth (about $2\pi/28$ days).

We then have

$$\left[\left(\frac{di}{dt}\right)_{moon} = -\frac{3}{4}\frac{1}{81.3}\frac{\left(360\frac{365}{28}\right)^2}{360\times 365}\sin 23.44^\circ \cos 23.44^\circ = -0.56^\circ/yr\right] \tag{16}$$

This is for the average γ_M in the 18.6 yr. lunar precession cycle.

At the peak of γ_M,

$$\left(\frac{di}{dt}\right)_{moon} = -0.65^\circ/yr,$$

and at the minimum,

$$\left(\frac{di}{dt}\right)_{moon} = -0.46^\circ/yr.$$

Adding these values to the solar rotation (both act in the same direction) yields

$$\boxed{\left(\frac{di}{dt}\right)_{Total} = -0.83^\circ/yr} \text{ (Maximum=-0.92, Minimum=-0.73)} \quad (17)$$

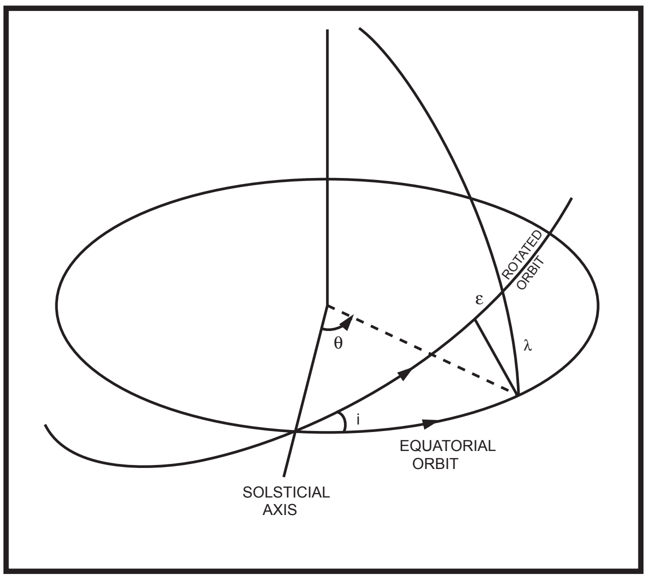
This can be easily translated into an equivalent ΔV:

$\Delta V = v_G \Delta i$;

$$v_G = \sqrt{\frac{\mu_E}{R_G}} = 3070 \text{ m/s}$$

$$\boxed{\Delta V = 44 \text{ m/s/yr}} \qquad \text{(Maximum=49, Minimum=39)} \quad (18)$$

What is the effect of i if left uncompensated?



Using some spherical trigonometry,

$$\sin\lambda = \frac{\sin i \sin\theta}{\sqrt{1 - \sin^2 i \cos^2\theta}} \quad (19)$$

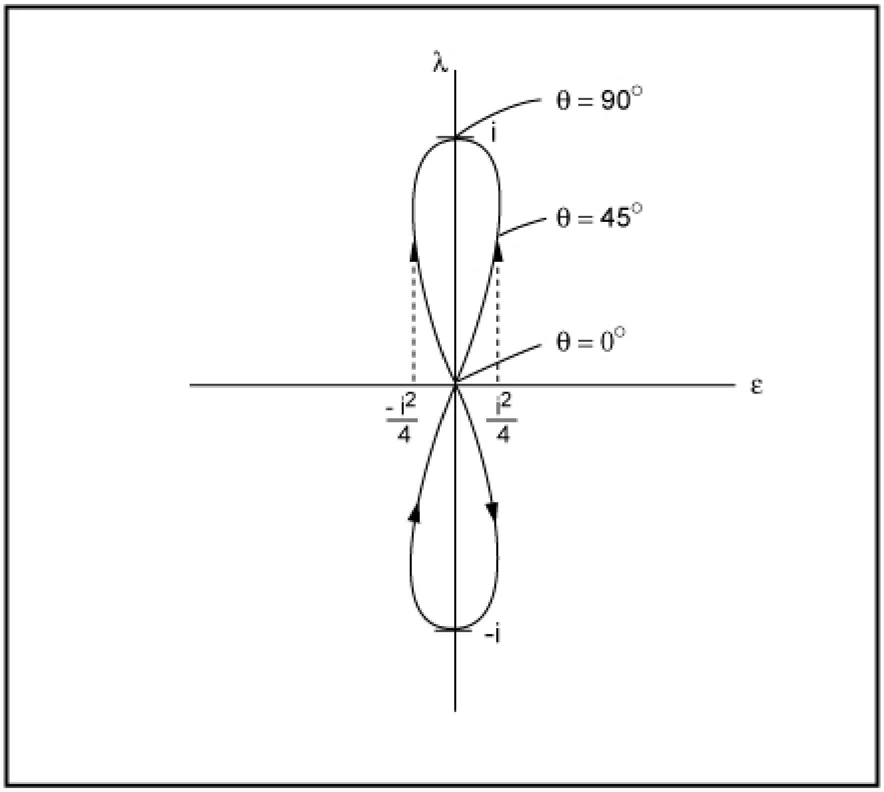
$$\sin\varepsilon = \frac{(1 - \cos i)\sin\theta\cos\theta}{\sqrt{1 - \sin^2 i \cos^2\theta}} \quad (20)$$

and for small rotations i,

$$\lambda \simeq i \sin\theta \quad (21)$$

$$\varepsilon \simeq \frac{i^2}{2}\sin\theta\cos\theta \quad (22)$$

So, during one day ($0<\theta<360^\circ$), the orbit describes, as seen from the ground, a figure 8 in the sky, about its nominal position:



The main deviation is λ (N/S direction), and this is called a "N/S drift". Typically, communications geosynchronous satellites can only tolerate $0.05^\circ - 0.1^\circ$ of such drift before correction is needed.