16.522, Space Propulsion
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**Lecture 2: Mission Analysis for Low Thrust**

1. Constant Power and Thrust: Prescribed Mission Time

Starting with a mass $M_0$, and operating for a time t an electric thruster of jet speed c, such as to accomplish an equivalent (force-free) velocity change of $\Delta V$, the final mass is

$$M\frac{dv}{dt} = -\frac{dM}{dt}c$$
$$dv = -c\frac{dM}{M}$$

if c=constant (consistent with constant power and thrust), then

$$v = c\ln\frac{M_0}{M_f}$$
$$M_f = M_0 e^{-\Delta V/c} \tag{1}$$

and the propellant mass used

$$M_P = M_0\left(1 - e^{-\frac{\Delta V}{c}}\right) \tag{2}$$

The structural mass is comprised of a part Mso which is independent of power level, plus a part $\alpha P$ proportional to rated power P, where $\alpha$ is the specific mass of the powerplant and thruster system. In turn, the power can be expressed as the rate of expenditure of jet kinetic energy, divided by the propulsive efficiency:

$$P = \frac{1}{2\eta}\dot{m}c^2 \tag{3}$$

and, since $\dot{m}$ is also a constant in this case,

$$\dot{m} = M_p/t.$$

Altogether, then,

$$M_s = M_{so} + \frac{\alpha}{2\eta}\frac{M_P}{t}c^2 \tag{4}$$

The payload mass is

$$M_L = M_f - M_s.$$

Combining the above expressions,

$$\frac{M_L}{M_o} = e^{-\Delta V/c} - \frac{M_{so}}{M_o} - \frac{\alpha c^2}{2\eta t}\left(1 - e^{-\Delta V/c}\right) \qquad (5)$$

Stuhlinger[1] introduced a "characteristic velocity"

$$v_{ch} = \sqrt{\frac{2\eta t}{\alpha}} \qquad (6)$$

whose meaning, from the definition of $\alpha$ is that, if the powerplant mass above were to be accelerated by converting all of the electrical energy generated during t, it would then reach the velocity $v_{ch}$.

Since other masses are also present, $v_{ch}$ must clearly represent an upper limit to the achievable mission $\Delta V$ and is in any case a convenient yardstick for both $\Delta V$ and c.

Figure 1 shows the shape of the curves of $\frac{M_L + M_{so}}{M_o}$ versus $c/v_{ch}$ with $\Delta V/v_{ch}$ as a parameter. The existence of an optimum c in each case is apparent from the figure. This optimum c is seen to be near $v_{ch}$ hence greater than $\Delta V$. If $\frac{\Delta V}{c}$ is taken to be a small quantity, expansion of the exponentials in (5) allows an approximate analytical expression for the optimum c:

$$c_{OPT} \cong v_{ch} - \frac{1}{2}\Delta V - \frac{1}{24}\frac{\Delta V^2}{v_{ch}} \qquad (7)$$

Figure 1 also shows that, as anticipated, the maximum $\Delta V$ for which a positive payload can be carried (with negligible $M_{so}$) is of the order of $0.8\, v_{ch}$. Even at this high $\Delta V$, Equation (7) is seen to still hold fairly well. To the same order of approximation, the mass breakdown for the optimum c is as shown in Figure 2.

The effects of (constant) efficiency, powerplant specific mass and mission time are all lumped into the parameter $v_{ch}$. Equation (7) then shows that a high specific impulse $I_{sp} = c/g$ is indicated when the powerplant is light and/or the mission is allowed a long duration. Figure 2 then shows that, for a fixed $\Delta V$, these same attributes tend to give a high payload fraction and small (and comparable) structural and fuel fractions. Of course the same breakdown trends can be realized by reducing $\Delta V$ for a fixed $v_{ch}$. This regime was called quite graphically the "trucking" regime by Loh [2]. At the opposite end (short mission, heavy powerplant) we have a low $v_{ch}$, hence low optimum specific impulse, and, from Figure 2, small payload and large fuel fractions. This is then the "sports car" regime [2].

References:

Ref. [1]: Stuhlinger, E. *Ion Propulsion For Space Flight.* New York: Mc Graw-Hill Book Co., 1964.

Ref. [2]: Loh, W. H. *Jet, Rocket, Nuclear, Ion and Electric Propulsion Theory and Design*. New York: Springer-Verlag, 1968.

We have, so far, regarded the efficiency $\eta$ as a constant, independent of the choice of specific impulse. This is not, in general, a good assumption for electric thrusters where the physics of the gas acceleration process can change significantly as the power loading (hence the jet velocity) is increased. For each thruster family (resistojets, arcjets, ion engines, MPD thrusters) and for each fuel and design, one can typically establish a connection between $\eta$ and c alone. Thus, as we will see in detail later, $\eta$ increases with c in both ion and MPD thrusters, whereas it typically decays with c for arcjets (beyond a certain c). In general, then, one needs to return to Equation (5) with $\eta = \eta(c)$ in order to discover the best choice of c in each case. It is instructive to consider in some detail the particular case of the ion engine, both because of its own importance and because relatively simple and accurate laws can be obtained in that case.

Ion engine losses can be fairly well characterized by a constant voltage drop per accelerated ion. If this is called $\Delta\phi$, and singly charged ions are assumed, the energy spent per ion is

$$1/2\, m_i c^2 + \Delta\phi \qquad \left(m_i = \text{ion mass;}\ \ e = \text{electron charge}\right),$$

of which only $1/2\, m_i\, c^2$ is useful.

The efficiency is then

$$\eta = \frac{c^2}{c^2 + \dfrac{2e\Delta\phi}{m_i}} \qquad (8)$$

We should also include a factor of $\eta_0 \angle 1$ to account for power processing and other losses. We then have

$$\eta = \eta_0 \frac{c^2}{c^2 + v_L^2} \qquad (9)$$

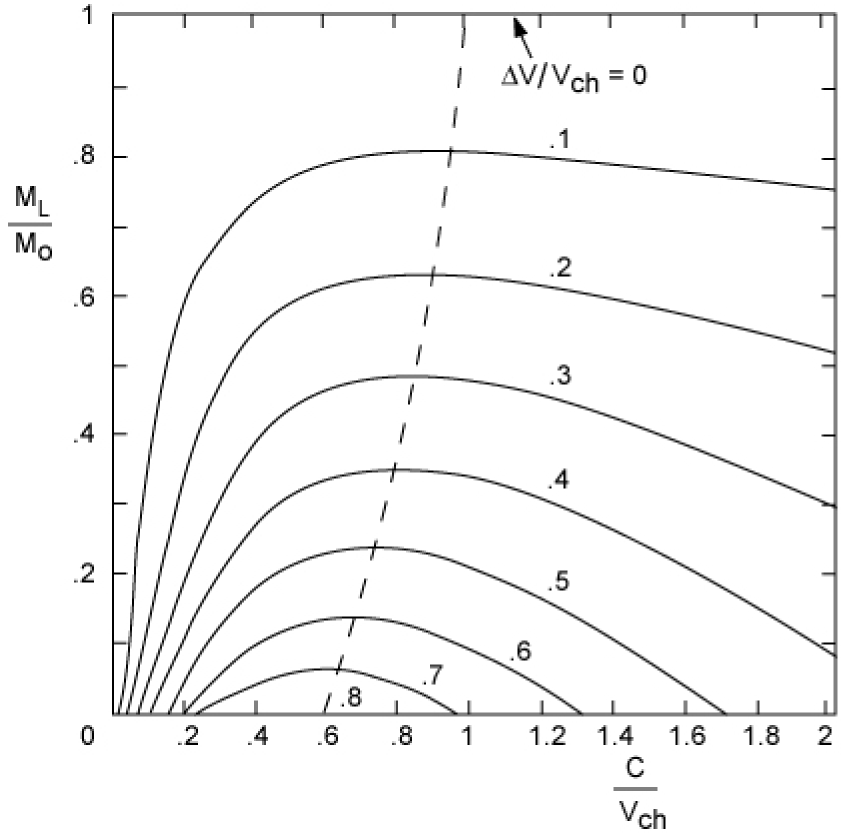



Fig. 1 Payload Fraction vs. Specific Impulse, Mission $\Delta V$ and Characteristic Velocity For $\Delta V \ll V_{ch}$

$c_{OPt} \simeq V_{ch} - \Delta V/2$; $(M_L/M_O)_{MAX} \simeq (1 - \Delta V/V_{ch})^2$

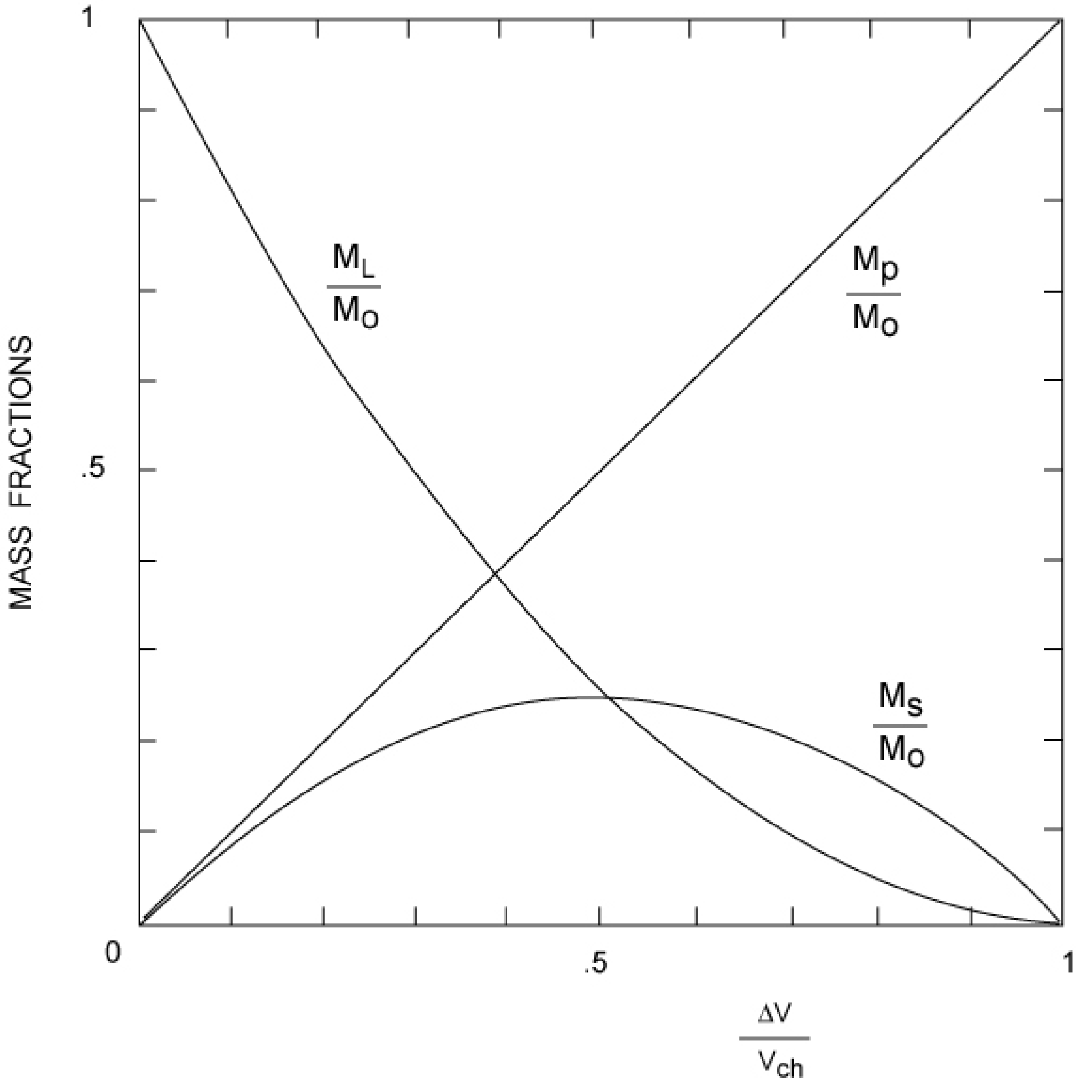



Fig. 2 Optimized Mass Fractions

were $v_L$ is a "loss velocity", equal to the velocity to which one ion would be accelerated by the voltage drop $\Delta\phi$. Notice how this simple expression already indicates the importance of a high atomic mass propellant; $\Delta\phi$ is insensitive to propellant choice, and so $v_L$ can be reduced if $m_i$ is large. Equation (9) also shows the rapid loss of efficiency when c is reduced below $v_L$.

Using (9), we can rewrite (5) as

$$\frac{M_L}{M_o} = e^{-\frac{\Delta V}{c}} - \frac{M_{so}}{M_o} - \frac{c^2 + v_L^2}{v_{ch}^2}\left(1 - e^{-\frac{\Delta V}{c}}\right) \quad (10)$$

where the definition of $v_{ch}$ (Equation (6)) is now made using $\eta_0$ instead of $\eta$. Once again, only approximate expressions for $\Delta V/v_{ch}$ are feasible for the optimum c and mass fractions. Normalizing all velocities by $v_{ch}$:

$$x \equiv \frac{c}{v_{ch}}; \; v = \frac{\Delta v}{v_{ch}}; \; \delta = \frac{v_L}{v_{ch}} \quad (11)$$

we obtain

$$x_{OPT} = \sqrt{1+\delta^2} - \frac{v}{2} - \frac{v^2}{24\sqrt{1+\delta^2}} + \ldots \quad (12)$$

$$\left(\frac{M_L}{M_o}\right)_{MAX} + \frac{M_{so}}{M_o} = 1 - 2\sqrt{1+\delta^2}\, v + v^2 - \frac{v^3}{12\sqrt{1+\delta^2}} + \ldots \quad (13)$$

$$\frac{M_P}{M_o} = \frac{v}{\sqrt{1+\delta^2}} - \frac{1}{24}\left(\frac{v}{\sqrt{1+\delta^2}}\right)^3 + \ldots \quad (14)$$

For $\delta = 0$, and neglecting the last term included in each case, we recover the simple expressions of Equation (7) and Figure 2. The main effect of the losses $(\delta)$ can be seen to be:

(a) An increase of the optimum c, seeking to take advantage of the higher efficiency thus obtained.
(b) A reduction of the maximum payload,
(c) A reduction of the fuel fraction.

Both these last effects indicate a higher structural fraction, due to the need to raise rated power to compensate for the efficiency loss. It is worth noting also that the losses are felt least in the "trucking" mode (high $v_{ch}$, i.e. light engine or long duration).

## 2. The Optimum: Thrust Profile

As was mentioned, there is no a priori reason to operate an electric thruster at a constant thrust or specific impulse, even if the power is indeed fixed. We examine here a simple case to illustrate this point, namely, one with a constant efficiency as in the classical Stuhlinger optimization, but allowing F, $\dot{m}$ and c to vary in time if this is advantageous. Of course these variations are linked by the constancy of the power:

$$P = \frac{1}{2\eta}\dot{m}(t)c^2(t) = \frac{1}{2\eta}F(t)c(t) \tag{15}$$

Consider the rate of change of the inverse mass with time:

$$\frac{d\left(\frac{1}{M}\right)}{dt} = -\frac{1}{M^2}\frac{dM}{dt} = \frac{\dot{m}}{M^2} \tag{16a}$$

Multiplying and dividing by $F^2 = \dot{m}^2 c^2$,

$$\frac{d\left(\frac{1}{M}\right)}{dt} = \frac{F^2}{M^2\dot{m}c^2} = \frac{a^2}{2\eta P} \tag{16b}$$

where $a = F/M$ is the acceleration due to thrust.

Integrating,

$$\frac{1}{M_f} - \frac{1}{M_0} = \frac{1}{2\eta P}\int_0^t a^2 dt \tag{17}$$

On the other hand, the mission $\Delta V$ is

$$\Delta V = \int_0^t a\, dt \tag{18}$$

and is a prescribed quantity. We wish to select the function a(t) which will give a maximum $M_f$ (Equation 17) while preserving this value of $\Delta V$. The problem reduces to finding the shape of a(t), whose square integrates to a <u>minimum</u> while its own value has a fixed integral. The solution (which can be found by various mathematical techniques, but is intuitively clear) is that <u>a should be a constant</u>.

Using this condition, (17) and (18) integrate immediately. Eliminating a between these, we obtain

$$\frac{M_f}{M_0} = \frac{1}{1 + \frac{M_0 \Delta V^2}{2\eta t P}} \tag{19}$$

The <u>level</u> of power is yet to be selected; it will determine the average specific impulse, and it is to be expected that an optimum will also exist.

Using

$$M_f = M_L + M_s$$

and

$$P = \frac{M_s}{\alpha}$$

and introducing the characteristic velocity (Equation 6), we rewrite (19) as

$$\frac{M_L}{M_0} = \frac{M_s}{M_0}\left[\frac{1}{\left(M_s/M_0\right) + \left(\Delta V/v_{ch}\right)^2} - 1\right] \qquad (20)$$

and select the value of $\frac{M_s}{M_0}$ that will maximize $\frac{M_L}{M_0}$. This is easily found to be

$$\left(\frac{M_s}{M_0}\right)_{OPT} = \frac{\Delta V}{v_{ch}}\left(1 - \frac{\Delta V}{v_{ch}}\right) \qquad (21)$$

which, when used back in (21) gives

$$\left(\frac{M_L}{M_0}\right)_{MAX} = \left(1 - \frac{\Delta V}{v_{ch}}\right)^2 \qquad (22)$$

and then

$$\left(\frac{M_P}{M_0}\right)_{OPT} = \frac{\Delta V}{v_{ch}} \qquad (23)$$

These are, within the assumptions, <u>exact</u> expressions. They are to be compared to the <u>approximate</u> expressions in Figure 2 or Equation (12)-(14) with $\delta = 0$, $\frac{M_{s0}}{M_0} = 0$ which were found to apply when c, and not a, was assumed constant. Clearly, the difference is noticeable only for $v = \Delta V/v_{ch}$ near unity (its highest value), and is negligible for smaller values.

It is of some interest to inquire at this point how the jet velocity c should vary with time in order to keep the acceleration constant.

We have

$$a = \frac{\dot{m}c}{M} = \frac{\dot{m}}{M^2}Mc = \frac{a^2}{2\eta P}Mc$$

where (16 a, b) have been used. Hence,

$$c = \frac{2\eta P}{a}\frac{1}{M}$$

and since, by (16b), $\frac{1}{M}$ varies <u>linearly</u> with time, so will c. At the final time, when $M = M_f$,

$$c_f = \frac{2\eta P}{aM_f} = \frac{2\eta M_s/\alpha}{\frac{\Delta V}{t}M_f} = \frac{v_{ch}^2}{\Delta V}\frac{M_s}{M_s + M_L}$$

and, from (21), (22),

$$\frac{M_s}{M_s + M_L} = \frac{\Delta V}{v_{ch}}$$

so that

$$c_f = v_{ch} \tag{24}$$

The rate of change of c follows from that of $1/M$ (Equation 16) as

$$\frac{dc}{dt} = \frac{2\eta P}{a}\frac{d\left(\frac{1}{M}\right)}{dt} = a \tag{25}$$

so that, altogether, if $t'$ represents some intermediate time, while t is the final time (used in $v_{ch}$),

$$c(t') = v_{ch} - a(t - t') \tag{26}$$

This varies between

$c = v_{ch} - \Delta V$ at $t' = 0$

and

$c = v_{ch}$ at $t' = t$.

The approximate result $c_{OPT} \cong v_{ch} - \frac{1}{2}\Delta V$ found when c was constrained to remain constant is therefore quite reasonable. Notice that (26) implies a constant <u>absolute</u> velocity of the exhaust gas, at the value

$$c_{abs} = v_{ch} - \Delta V - V(0)$$

<u>Alternative Derivation with Variable Specific Impulse</u>

This is a more general treatment than that in pp. 15-19 of the Notes, and can be extended to more complicated situations, like non-constant efficiency. It also introduces some elements of Calculus of Variations, which is of general utility, and has many commonalities with Optimal Control theory.

We wish to minimize

$$\Delta V = \int_0^{t_f} \left(\frac{F}{m}\right) dt \qquad (1)$$

subject to a given power (constant in time)

$$p = \frac{1}{2}\frac{\dot{m}\,c^2}{\eta} = \frac{1}{2}\frac{Fc}{\eta} \qquad (2)$$

and to

$$\dot{m} = -\frac{dm}{dt} \qquad (3)$$

write (1) as

$$\Delta V = \int_0^{t_f} \frac{2\eta P}{mc}\, dt \qquad (4)$$

which eliminates thrust.

Next, treat (3) as a dynamic <u>constraint</u>, and append it to the cost through a time-dependent Lagrange multiplier $\lambda(t)$.

Define the Hamiltonian

$$H = \frac{2\eta P}{mc} - \lambda\left(\dot{m} + \frac{dm}{dt}\right) \qquad (5)$$

or, using (2),

$$H = \frac{2\eta P}{mc} - \lambda\left(\frac{2\eta P}{c^2} + \frac{dm}{dt}\right) \qquad (6)$$

and minimize (unconstrained) the integral $\int_0^{t_f} Hdt$ .To do this, perturb about the optimum solution:

$$\delta\int_0^{t_f} Hdt = \int_0^{t_f}\left[\frac{\partial H}{\partial c}\delta c + \frac{\partial H}{\partial m}\delta m + \frac{\partial H}{\partial\left(\frac{dm}{dt}\right)}\underbrace{\delta\left(\frac{dm}{dt}\right)}_{\frac{d}{dt}(\delta m)}\right]dt = 0$$

Integrate last term by parts:

$$\int_0^{t_f}\left[\frac{\partial H}{\partial c}\delta c+\left(\frac{\partial H}{\partial m}-\frac{d}{dt}\left(\frac{\partial H}{\partial\left(\frac{dm}{dt}\right)}\right)\right)\delta m\right]dt+\left[\frac{\partial H}{\partial\left(\frac{dm}{dt}\right)}\delta m\right]_0^{t_f}=0 \qquad (17)$$

For optimality, we want to impose

$$\frac{\partial H}{\partial c}=0 \qquad (18)$$

$$\frac{\partial H}{\partial m}=\frac{d}{dt}\left(\frac{\partial H}{\partial\frac{dm}{dt}}\right) \qquad (19)$$

and at the ends, we say

$m(t)=m_0$ (fixed),

so

$(\delta m)_0=0$,

and also

$m(t_f)=m_f$ (fixed, since we minimize $\Delta V$ between given masses),

so, again,

$(\delta m)_{t_f}=0$.

Equation (18) gives in general (assuming $\eta=\eta(c)$)

$$-\frac{2P\eta}{mc^2}+2\lambda\frac{2\eta P}{c^3}+\left(\frac{2P}{mc}-\lambda\frac{2P}{c^2}\right)\frac{\partial\eta}{\partial c}=0$$

or

$$-\frac{1}{m}+\frac{2\lambda}{c}+\left(\frac{c}{m}-\lambda\right)\frac{\partial\ln\eta}{\partial c}=0 \qquad (20)$$

For example, if

$$\eta=\eta_0\frac{c^2}{c^2+v_2^2},$$

$$\frac{\partial \ln \eta}{\partial c} = \frac{2v_2^2}{c\left(c^2 + v_2^2\right)}$$

Here we take the simple case where $\eta$ = constant, so (20) gives

$$c = 2\lambda m \tag{21}$$

From (19),

$$-\frac{2\eta P}{m^2 c} + \frac{d\lambda}{dt} = 0 \tag{22}$$

and using (21),

$$\frac{d\lambda}{dt} = \frac{\eta P}{\lambda m^3} \tag{24}$$

We also can substitute (21) into (3) to get

$$\frac{dm}{dt} = -\frac{\eta P}{2\lambda^2 m^2} \tag{25}$$

Divide (24) by (25):

$$\frac{d\lambda}{dm} = -2\frac{\lambda}{m} \tag{26}$$

which integrates to

$$\lambda m^2 = A \tag{27}$$

(A = undetermined <u>constant</u>). The value of A can be easily related to the optimized $\Delta V$ from Equation (4):

$$\Delta V = \int_0^{t_f} \frac{2\eta P}{mc}\, dt = \int_0^{t_f} \frac{\eta P}{\lambda m^2}\, dt = \frac{\eta P}{A} t_f$$

$$\therefore A = \frac{\eta P t_f}{\Delta V} \tag{28}$$

To complete the time integration, go back to (24):

$$\frac{d\lambda}{dt} = \frac{\eta P}{\lambda\left(\frac{A}{\lambda}\right)^{3/2}} = \frac{\eta P}{A^{3/2}} \lambda^{1/2}\ ;$$

$$\lambda^{-1/2}\, d\lambda = \frac{\eta P}{A^{3/2}}\, dt$$

and integrating,

$$2\lambda^{1/2} = \frac{\eta P}{2A^{3/2}}t + B$$

$$\lambda = \left(\frac{\eta P}{2A^{3/2}}t + \frac{B}{2}\right)^2 \tag{29}$$

and from (27),

$$m = \frac{A^{1/2}}{\frac{\eta P}{2A^{3/2}}t + \frac{B}{2}} \tag{30}$$

The constants A and B can now be related to $m_0$ and $m_f$:

$$m_0 = \frac{2A^{1/2}}{B}; \qquad m_f = \frac{A^{1/2}}{\frac{\eta P t_f}{2A^{3/2}} + \frac{B}{2}} = \frac{1}{\frac{\eta P t_f}{2A^2} + \frac{1}{m_0}} = \frac{m_0}{1 + \frac{m_0 \eta P t_f}{2A^2}} \tag{31}$$

Using now (28),

$$\frac{m_0 \eta P t_f}{2A^2} = \frac{m_0 \eta P t_f}{2\left(\frac{\eta P t_f}{\Delta V}\right)^2} = \frac{m_0 \Delta V^2}{2\eta P t_f} \tag{32}$$

We now introduce the specific mass of the power/propulsion system $\alpha \equiv \frac{M_{PP}}{P}$, and the Characteristic Velocity

$$v_{ch} = \sqrt{\frac{2\eta t_f}{\alpha}},$$

to rewrite (32) as

$$\frac{m_0 \eta P t_f}{2A^2} = \frac{m_0}{m_{pp}}\left(\frac{\Delta V}{v_{ch}}\right)^2$$

We also remember now that the final mass $m_f$ contains payload $m_L$, power/propulsion mass, and random dry mass $m_{so}$, and write (31) as

$$m_L + m_{so} + m_{pp} = \frac{m_0}{1 + \frac{m_0}{m_{pp}}\left(\frac{\Delta V}{v_{ch}}\right)^2} \tag{33}$$

or

$$\frac{m_L + m_{so}}{m_0} = \frac{m_{pp}}{m_0}\left[\frac{1}{\frac{m_{pp}}{m_0} + \left(\frac{\Delta V}{v_{ch}}\right)^2} - 1\right] \tag{34}$$

At this point we have the optimum time profiles, but we can do better by <u>also</u> selecting the <u>optimum power</u> level, which amounts to selecting the <u>optimum specific impulse</u> as well. We allow $x = \frac{m_{pp}}{m_0}$ to vary in (34) and maximize $\frac{m_L + m_{so}}{m_0} = y$.

Using

$$\frac{\Delta V}{v_{ch}} = v \tag{35}$$

We had

$$y = \frac{x}{x + v^2} - x$$

$$\frac{dy}{dx} = \frac{x + v^2 - x}{\left(x + v^2\right)^2} - 1 = 0$$

$$v^{\cancel{2}} = \left(x + v^2\right)^{\cancel{2}}$$

$$x = v - v^2 = v\left(1 - v\right)$$

or

$$\boxed{\left(\frac{m_{pp}}{m_0}\right)_{OPT} = \frac{\Delta V}{v_{ch}}\left(1 - \frac{\Delta V}{v_{ch}}\right)} \tag{36}$$

Putting $x = v - v^2$ into $y$,

$$y_{OPT} = \frac{v - v^2}{v} - v + v^2 = v\left(1 - v\right)\left(\frac{1}{v} - 1\right)$$

$$y_{OPT} = \left(1 - v\right)^2$$

or

$$\boxed{\left(\frac{m_L + m_{so}}{m_0}\right)_{OPT} = \left(1 - \frac{\Delta V}{v_{ch}}\right)^2} \tag{37}$$

and then the propellant fraction follows from

$$\frac{m_p}{m_0} = 1 - \frac{m_L + m_{S0}}{m_0} - \frac{m_{pp}}{m_0}$$

$$\boxed{\left(\frac{m_p}{m_0}\right)_{OPT} = \frac{\Delta V}{v_{ch}}} \qquad (38)$$

Note these are exact formulas, no trailing terms. The specific impulse now follows from (21):

$$c = 2\lambda m = \frac{2A}{m} = \frac{2\eta P t_f}{m_0 \Delta V}\ \frac{m_0}{m} = \frac{2\eta P t_f}{m_0 \Delta V}\left(1 + \frac{m_0 \Delta V^2 t}{2\eta P t_f^2}\right)$$

$$c = \frac{2\eta P t_f}{m_0 \Delta V} + \frac{t}{t_f}\Delta V$$

and if we use optimum power,

$$c = \frac{m_{pp}}{m_0}\frac{v_{ch}^2}{\Delta V} + \frac{t}{t_f}\Delta V = v_{ch}\left(1 - \frac{\Delta V}{v_{ch}}\right) + \frac{t}{t_f}\Delta V$$

$$\boxed{c = v_{ch} - \Delta V\left(1 - \frac{t}{t_f}\right)} \qquad (39)$$

so that c increases linearly from $V_{ch}$ at $t = 0$ to $V_{ch} - \Delta V$ at $t_f$.

Optimum Mission Time

So far, t has been a free parameter, and we have found the mission performance (payload fraction) to improve with large t. But time has some costs associated with it, so increasing t is not necessarily desirable. These "costs" of time including capital immobilization, personnel costs during the long thrusting period, loss of opportunity, etc. There are several simple analytical ways to penalize long t choices. We select here one that maximizes the "Transportation Rate" $M_L/t$. This makes most sense when there is a sequence of identical flights, each delivering payload $M_L$ in time t, but it can serve as a crude indicator even for a single mission.

We also assume constant thrust, constant power (for simplicity of operation), and we include in the analysis the effect of a variable efficiency

$$\eta = \frac{\eta_0}{\left(1 + c^2/v_L^2\right)},$$

with $v_L = \sqrt{2e\Delta v_{LOSS}/m_i}$, as in ion engines with constant ion cost $\Delta v_{LOSS}$.

We had for this case a payload ratio (optimized with respect to specific impulse)

$$\frac{M_L}{M_o} = 1 - 2\frac{\Delta v}{v_{ch}}\sqrt{1 + \left(\frac{v_L}{v_{ch}}\right)^2} + \left(\frac{\Delta v}{v_{ch}}\right)^2 \qquad (1)$$

where

$v_{ch} = \sqrt{\frac{2\eta_0 t}{\alpha}}$ ( $\alpha$ =specific mass of the Power/Propulsion system)

The thrusting time t is buried in $v_{ch}$. To bring it out more explicitly, let us define a reference time

$$t^* = \frac{\alpha\,\Delta V^2}{2\eta_0} \qquad (2)$$

which depends on specified quantities only, and then a non-dimensional time

$$\tau \equiv \frac{t}{t^*} \qquad (3)$$

From the definition of $v_{ch}$, then,

$$\tau = \frac{t}{t^*} = \frac{2\eta_0 t}{\alpha\,\Delta V^2} = \left(\frac{v_{ch}}{\Delta V}\right)^2 \qquad (4)$$

and so (1) becomes

$$\frac{M_L}{M_o} = 1 - \frac{2}{\sqrt{\tau}}\sqrt{1 + \frac{z^2}{\tau}} + \frac{1}{\tau} \quad ; \quad z \equiv \frac{v_L}{\Delta V} \qquad (5)$$

A normalized transportation rate is now defined as

$$\psi \equiv \frac{M_L/M_O}{t/t^*} \qquad (6)$$

or, from (5),

$$\psi = \frac{1}{\tau} - \frac{2}{\tau^{3/2}}\sqrt{1 + \frac{z^2}{\tau}} + \frac{1}{\tau^2} \qquad (7)$$

To maximize $\psi$, set $\frac{\partial \psi}{\partial\left(\frac{1}{\tau}\right)} = 0$.

After some re-grouping, this leads to the equation for $\tau_{OPT}$ :

$$\boxed{(\tau + 2)\sqrt{\tau + z^2} = 3\tau + 4z^2} \qquad (8)$$

A convenient way to view this $\tau_{OPT}(z)$ dependence is to define an intermediate parameter u= $\sqrt{\tau + z^2}$ ; we then obtain the parametric representation (for $u \geq 2$)

$$\begin{cases} \tau_{OPT} = \dfrac{2u(2u-1)}{1+u} \\ z = \sqrt{u^2 - \tau_{OPT}} \end{cases} \qquad (9)$$

A good analytical approximation (valid for z=0, asymptotic for z >>1) is

$$\tau_{OPT} \simeq 2 + 4z + \frac{6}{z+3} \qquad (10)$$

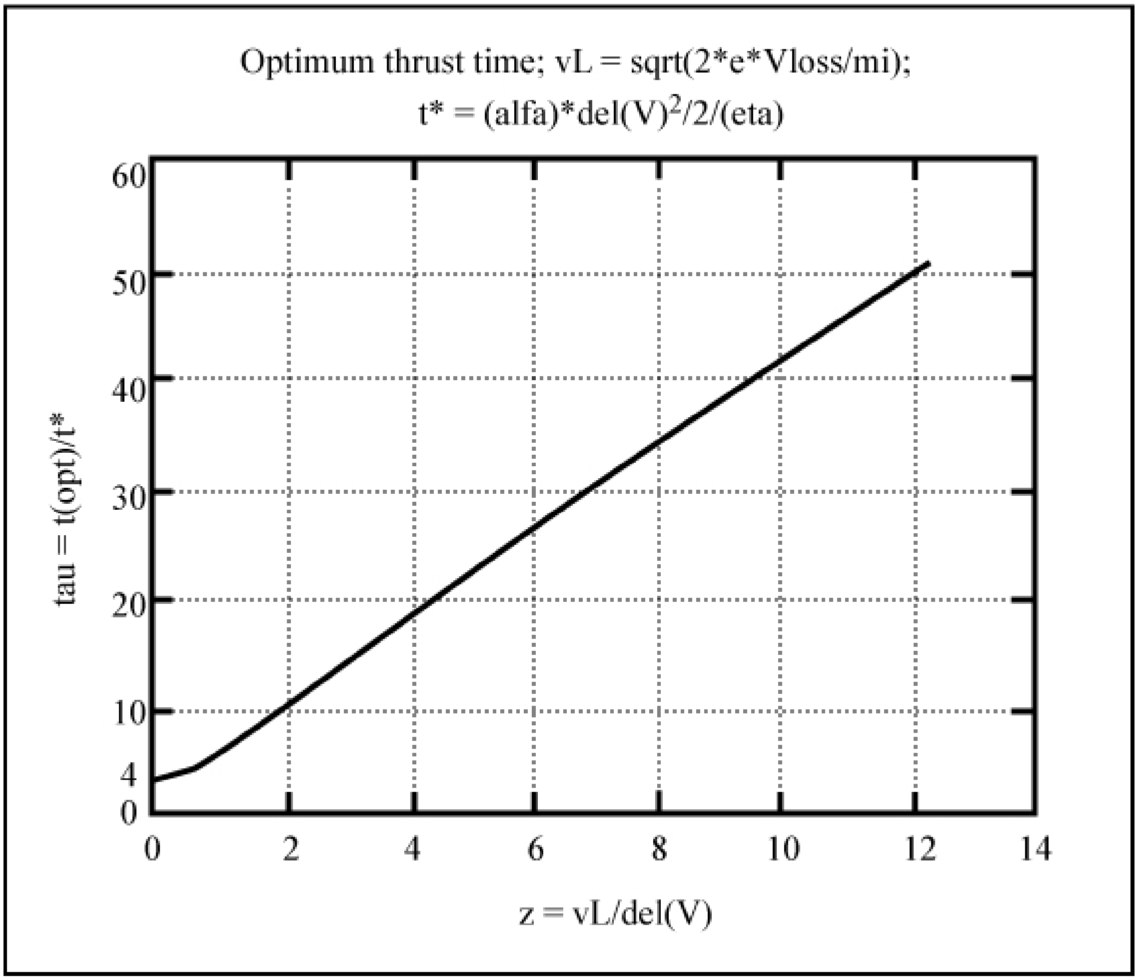
In particular, for a <u>constant-efficiency</u> model ($v_L$=0, z=0) we see that

$$\tau_{OPT} = 4,$$

or

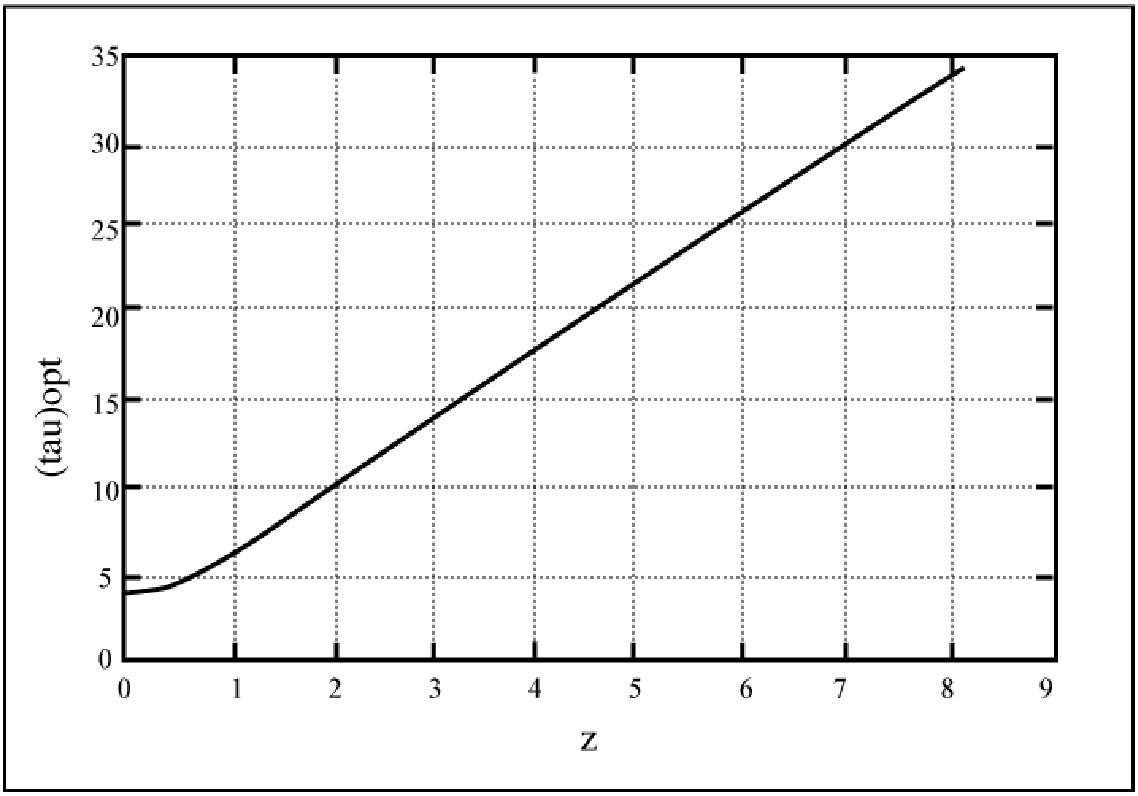
$$t_{OPT} = \frac{2\alpha\, \Delta V^2}{\eta_0}.$$

For other values of $v_L$ (or z), the results are shown in the attached graphs.

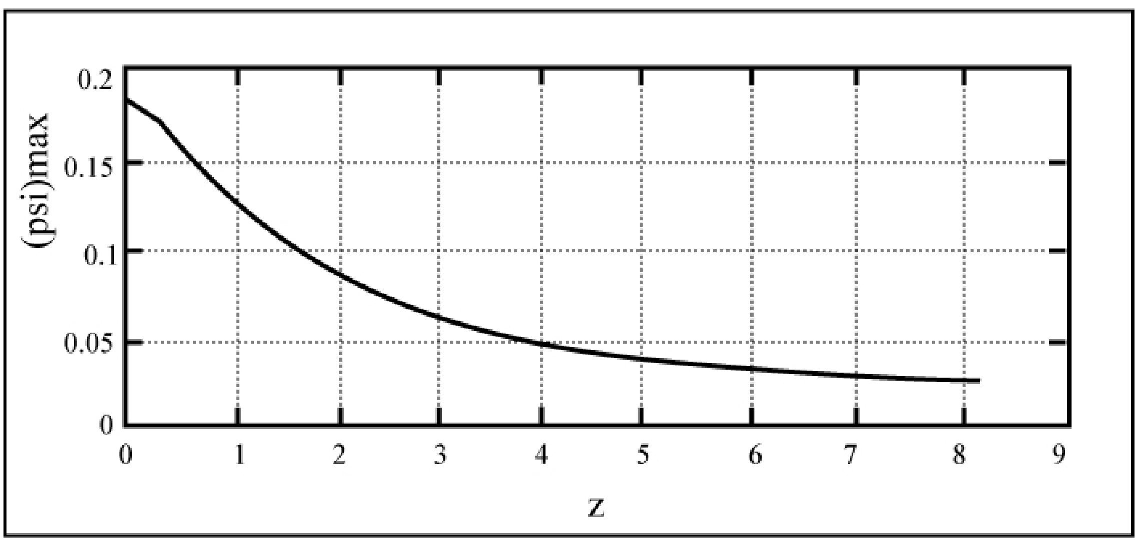



$$(\tau+2)\sqrt{\tau+z^2} = 3\tau+4z^2$$

$$\begin{cases} \tau \simeq 4z+2+\dfrac{6}{z+3}+\ldots & (z>>1) \\ \tau \simeq 4+5z^2+\ldots & (z<<1) \end{cases}$$



35
30
25
20
15
10
5
0
(tau)opt
0
1
2
3
4
5
6
7
8
9
z

0.2
0.15
0.1
0.05
0
(psi)max
0
1
2
3
4
5
6
7
8
9
z