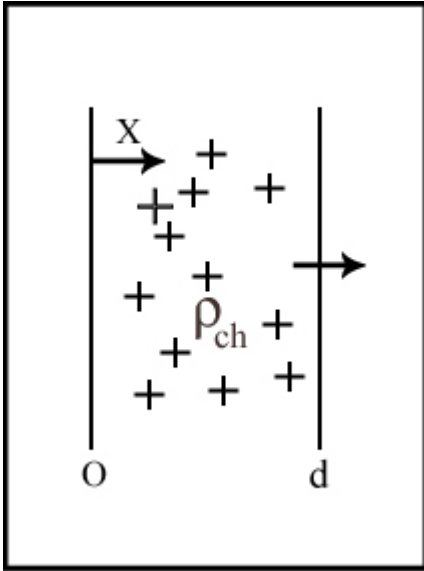
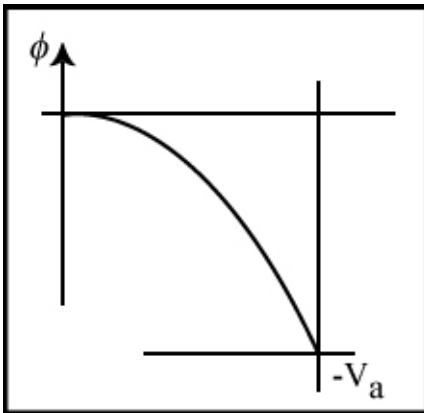


Lecture 15: Thrust Calculation (Single Grid, Single Potential)



$$\frac{F_x}{A} = \int_0^d \rho_{ch} E_x dx = \int_0^d \epsilon_0 \frac{dE_x}{dx} E_x dx = \frac{\epsilon_0 E_x^2}{2} \Big|_0^d$$



$$\phi = -V_a \left(\frac{x}{d}\right)^{4/3} \quad E_x = \frac{V_a}{d} \frac{4}{3} \left(\frac{x}{d}\right)^{1/3}$$

$$E_x|_0 = 0 \quad E_x|_d = \frac{4}{3} \frac{V_a}{d}$$

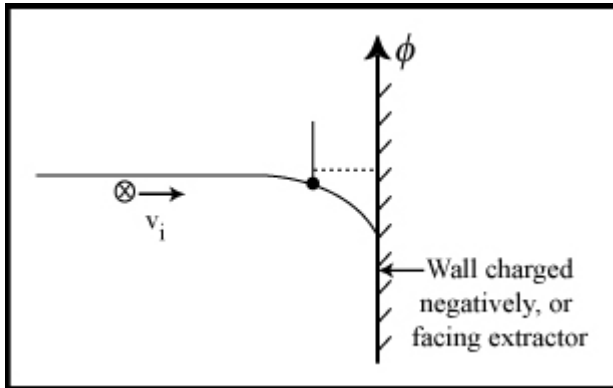
$$\left[\frac{F_x}{A} = \frac{\epsilon_0}{2} \frac{16}{9} \frac{V_a^2}{d^2} = \frac{8}{9} \epsilon_0 \frac{V_a^2}{d^2} \right]$$

Alternative:

$$\frac{F_x}{A} = \underbrace{\frac{m_i}{e}}_m j_c = \frac{m_i}{e} \frac{4\sqrt{2}}{9} \epsilon_0 \sqrt{\frac{e}{m_i} \frac{V_a^{3/2}}{d^2}} \sqrt{\frac{2eV_a}{m_i}} = \frac{8}{9} \epsilon_0 \frac{V_a^{3/2}}{d^2}$$

j from
Child-Langmuir

Bohm velocity: Why?



collisional "drag"

$$n_i m_i v_i \frac{dv_i}{dx} + \frac{dP_i}{dx} = en_i E_x - F_{in}$$

$$n_e m_e v_e \frac{dv_e}{dx} + \frac{dP_e}{dx} = -en_e E_x - F_{en}$$

small small

not near wall, $n_e = n_i$.

Add:

constant

$$n_e m_i v_i \frac{dv_i}{dx} + \frac{d(P_e + P_i)}{dx} \approx -F_{in}$$

and

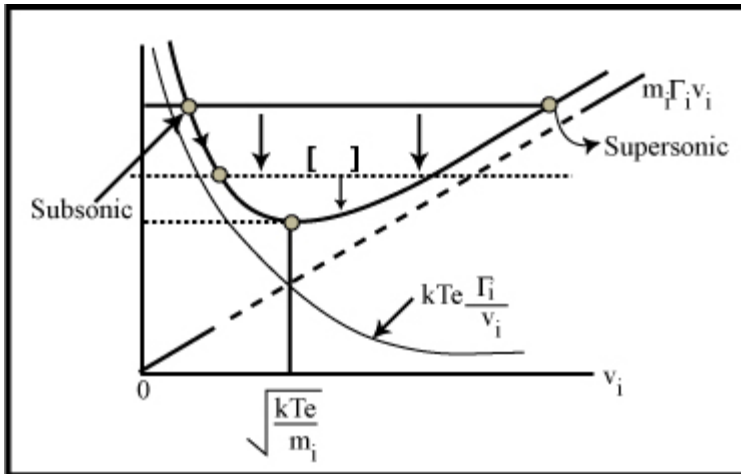
$$n_e v_i = \text{const} = \Gamma_i$$

$$\frac{d}{dx} (n_e m_i v_i^2 + P_e + P_i) \approx -F_{in}$$

$$P_e + P_i = n_e k (T_e + T_i) \approx n_e k T_e$$

$$\frac{d}{dx} \left[m_i \Gamma_i v_i + k T_e \frac{\Gamma_i}{v_i} \right] = -F_{in}$$

min. at $v_i = \sqrt{\frac{kT_e}{m_i}}$



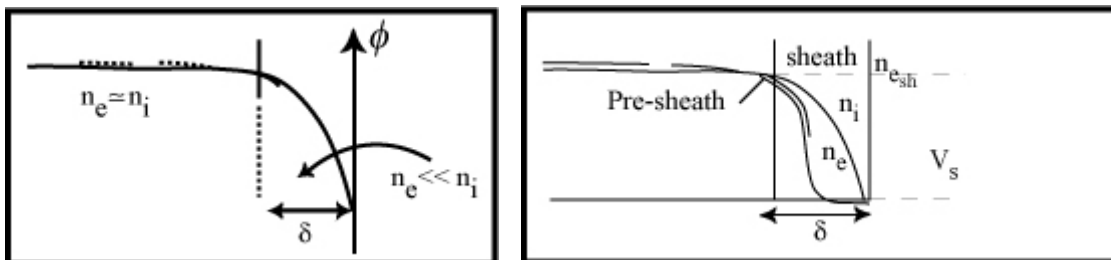
The lines $kT_e \frac{\Gamma_i}{v_i}$ and $m_i \Gamma_i v_i$ must cross at $v_i = \sqrt{\frac{kT_e}{m_i}}$, where their sum is minimum.

So, no solution at

$$v_i > \sqrt{\frac{kT_e}{m_i}} = v_B$$

Ions accelerate to $v_i = v_B$ in the quasineutral plasma. Beyond that, $n_e \ll n_i$, E_x becomes very strong, and ions just free-fall to wall (in the sheath) so, entering sheath, $v_i = v_B$.

How big is the sheath?



In sheath, say $n_e \approx 0$

$$\text{Child-Langmuir: } j_i = en_i v_i = \frac{4\sqrt{2}}{9} \epsilon_0 \sqrt{\frac{e}{m_i}} \frac{V_s^{3/2}}{\delta^2}$$

But also

$$j_i \approx en_{e_{sh}} \sqrt{\frac{kT_e}{m_i}}$$

$$\frac{4\sqrt{2}}{9} \epsilon_0 \sqrt{\frac{e}{m_i}} \frac{V_s^{3/2}}{\delta^2} \approx en_{e_{sh}} \sqrt{\frac{kT_e}{m_i}}$$

$$\delta^2 \approx \frac{4\sqrt{2}}{9} \frac{\epsilon_0 V_s^{3/2}}{\sqrt{ekT_e} n_{e_{sh}}} \frac{1}{n_{e_{sh}}} = \frac{4\sqrt{2}}{9} \frac{\epsilon_0 kT_e}{e^2 n_{e_{sh}}} \left(\frac{eV_s}{kT_e} \right)^{3/2}$$

$$n_{e_{sh}} = n_{e_{\infty}} \exp\left(\frac{-1}{2}\right)$$

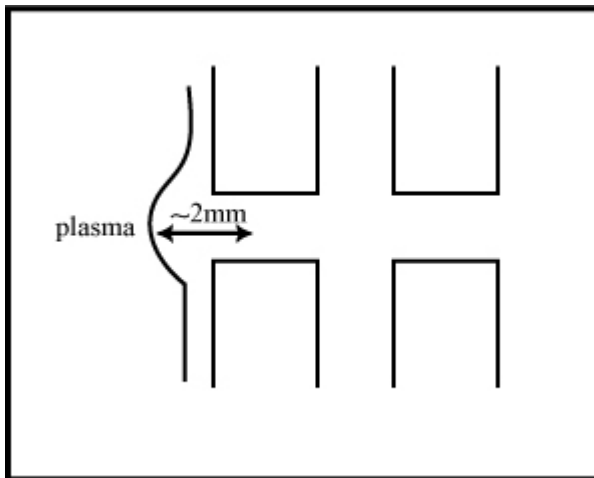
$$\delta \approx \underbrace{\left[\frac{4\sqrt{2}}{9e^{-1/2}} \right]^{1/2}}_{1.018} \underbrace{\left[\frac{\epsilon_0 kT_e}{e^2 n_{e_{\infty}}} \left(\frac{eV_s}{kT_e} \right)^{3/4} \right]^{1/2}}_{d_{\text{Debye}}}$$

$$d_{\text{Debye}} = 69 \sqrt{\frac{T_e \text{ (K)}}{n_e \text{ (m}^{-3}\text{)}}} \sim 69 \sqrt{\frac{3 \times 11600}{3 \times 10^{17}}} = 2.4 \times 10^{-5} \text{ m} = 24 \mu\text{m}$$

If wall not biased (insulator),

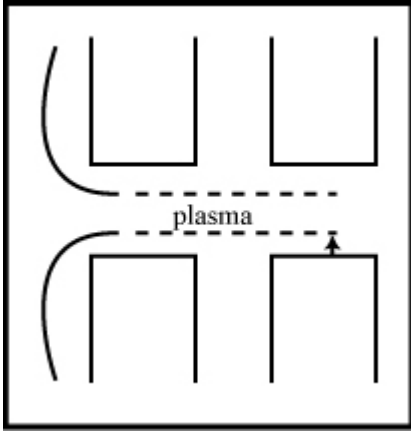
$$eV_s \sim kT_e \left(\ln \sqrt{\frac{m_i}{m_e}} \right) \rightarrow \delta \approx 5^{3/4} d_b \sim 3d_b$$

$$\text{For sheath in front of extractor grid, } \left. \begin{array}{l} V_s \sim 1000\text{V} \\ \frac{kT_e}{e} \sim 3\text{V} \end{array} \right\} \begin{array}{l} \delta \approx 78d_b \\ \delta \approx 1.9 \text{ mm} \end{array}$$

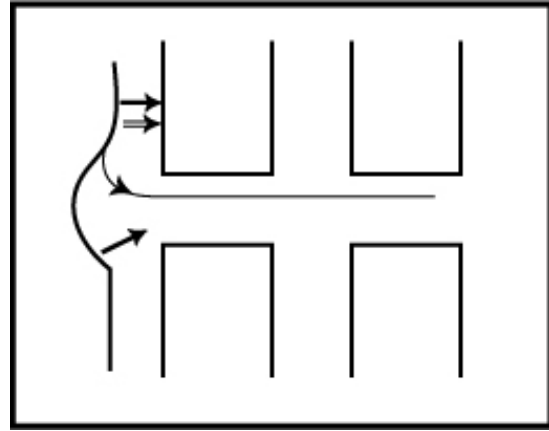


This approximately sizes the extractor holes.

If holes much bigger than δ , plasma would escape

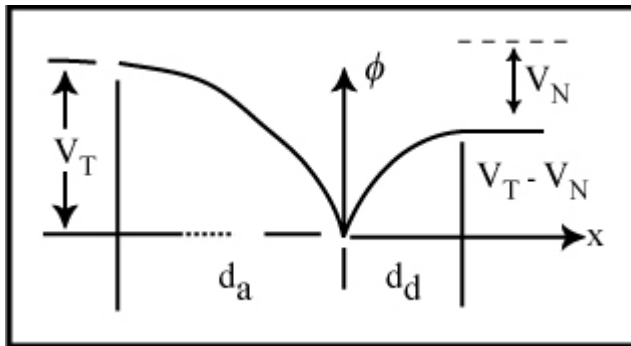


If much smaller, ions lost to grid



Space charge effects in the accel-decel gap

For $0 < x < d_d$,



$$\frac{d^2\phi}{dx^2} = -\frac{en_i}{\epsilon_0} = -\frac{j}{\epsilon_0 v_i} = -\frac{j}{\epsilon_0 \sqrt{v_0^2 - \frac{2e\phi}{m_i}}}$$

$$\frac{1}{2} \left(\frac{d\phi}{dx} \right)^2 = -\frac{j}{\epsilon_0} \int_0^\phi \frac{d\phi}{\sqrt{v_0^2 - \frac{2e\phi}{m_i}}} + c$$

Note: Slope here not necessarily zero.

Change integration variable :

$$v = \sqrt{v_0^2 - \frac{2e\phi}{m_i}}$$

$$\phi = \frac{m_i}{2e} (v_0^2 - v^2)$$

$$d\phi = -\frac{m_i}{e} v dv$$

$$\frac{1}{2} \left(\frac{d\phi}{dx} \right)^2 = c - \frac{j}{\epsilon_0} \int_{v_0}^v \frac{-\frac{m_i}{e} v dv}{\cancel{v}} = c - \frac{j}{\epsilon_0} \frac{m_i}{e} (v_0 - v) = c - \frac{j}{\epsilon_0} \frac{m_i}{e} \left(v_0 - \sqrt{v_0^2 - \frac{2e\phi}{m}} \right)$$

where

$$v_0^2 = \frac{2eV_T}{m_i}$$

$$\frac{1}{2} \left(\frac{d\phi}{dx} \right)^2 = 2c - \frac{2j}{\epsilon_0} \frac{m_i}{e} \left(\sqrt{\frac{2eV_T}{m_i}} - \sqrt{\frac{2e(V_T - \phi)}{m_i}} \right)$$

$$dx = \frac{d\phi}{\sqrt{2c - \frac{2j}{\epsilon_0} \frac{m_i}{e} \left(\sqrt{\frac{2eV_T}{m_i}} - \sqrt{\frac{2e(V_T - \phi)}{m_i}} \right)}}$$

$$d_d = \int_0^{\phi=V_T-V_N} \frac{d\phi}{\sqrt{2c - \frac{2j}{\epsilon_0} \frac{m_i}{e} \left(\sqrt{\frac{2eV_T}{m_i}} - \sqrt{\frac{2e(V_T - \phi)}{m_i}} \right)}} \Rightarrow c, \text{ then all profiles follows.}$$

In the limit when the second gap becomes choked as well, (as in the case with no decel grid, in which case d_d is the downstream sheath thickness)

$$\left(\frac{d\phi}{dx} \right)_{x=d_d} = 0 \Rightarrow c = \frac{j m_i}{\epsilon_0 e} \left(\sqrt{\frac{2eV_T}{m_i}} - \sqrt{\frac{2eV_N}{m_i}} \right)$$

$(\phi = V_T - V_N)$

Then

$$d_d = \int_0^{V_T-V_N} \frac{d\phi}{\sqrt{\frac{2j m_i}{\epsilon_0 e} \left(\sqrt{\frac{2e(V_T - \phi)}{m_i}} - \sqrt{\frac{2eV_N}{m_i}} \right)}}$$

Change again variable:

$$\sqrt{\frac{2e(V_T - \phi)}{m_i}} = v$$

$$d\phi = -\frac{m_i}{e} v dv$$

Appendix B
ELECTRON DIFFUSION IN A MAGNETIC FIELD

1) No \vec{B} Field

In electron momentum balance, main forces are pressure gradient and collisional retardation (no inertia):

$$\nabla P_e \simeq -n_e m_e \bar{v}_e \nu_e \quad (1)$$

Also

$$P_e = n_e k T_e,$$

$$\nabla P_e \simeq k T_e \nabla n_e.$$

Solve for flux:

$$\boxed{n_e \bar{v}_e = -\frac{k T_e}{m_e \nu_e} \nabla n_e} \quad (2)$$

This is Fick's law of diffusion

$$n_e \bar{v}_e = -D_e \nabla n_e, \text{ with a diffusivity}$$

$$\boxed{D_e = \frac{k T_e}{m_e \nu_e}} \quad (3)$$

$$(\nu_e = \sum n_j \bar{c}_e Q_{ej}, \text{ collision frequency})$$

2) With \vec{B} (\perp to ∇P_e)

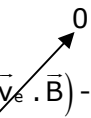
Add magnetic force:

$$\nabla P_e = -n_e m_e \bar{v}_e \nu_e - en_e \bar{v}_e \times \vec{B} \quad (4)$$

To solve for $n_e \bar{v}_e$, form

$$\nabla P_e \times \vec{B} = -m_e \nu_e n_e \bar{v}_e \times \vec{B} - en_e (\bar{v}_e \times \vec{B}) \times \vec{B},$$

and use

$$(\bar{v}_e \times \vec{B}) \times \vec{B} = \vec{B} (\bar{v}_e \cdot \vec{B}) - B^2 \bar{v}_e.$$


Eliminate $(\bar{v}_e \times \vec{B})$ between these two equations, simplify:

$$n_e \bar{v}_e = \frac{-\frac{kT_e}{m_e v_e} \nabla n_e + \frac{ekT_e}{m_e^2 v_e^2} \nabla n_e \times \bar{B}}{1 + \left(\frac{eB}{m_e v_e}\right)^2} \quad (5)$$

NOTE: This leaves the \bar{E} field out. To include it, just replace ∇n_e by $\nabla n_e + \frac{en_e \bar{E}}{kT_e}$

Define the nondimensional factor

$$\beta \equiv \frac{eB}{m_e v_e} \equiv \frac{\omega_c}{v_e} \quad (\text{Hall parameter}) \quad (6)$$

where $\omega_c = \frac{eB}{m_e}$ is the cyclotron frequency for electrons.

Then

$$n_e \bar{v}_e = \frac{1}{1 + \beta^2} (-D_e \nabla n_e - \bar{\beta} \times D_e \nabla n_e) \quad (7)$$

Of these two terms, the second is perpendicular to both, \bar{B} and ∇n_e , and is called the " $\nabla P_e \times \bar{B}$ drift". The main interest is on the first term, which is along $-\nabla n_e$, as a regular diffusion.

We see that this "cross-field diffusion" is governed by $n_e \bar{v}_e = -D_{\perp} \nabla n_e$, with

$$D_{\perp} = \frac{D_e}{1 + \beta^2} \quad (8)$$

So, a high Hall parameter β can greatly reduce diffusion, compared to that in the absence of a magnetic field. High β means both, high B and/or low collision frequency.

In an ion engine, with $T_e = 4\text{eV} = 46400\text{ K}$, the e-n and e-i cross-section are roughly

$$Q_{en} \approx 10^{-19} \text{ m}^2,$$

$$Q_{ei} \approx 4 \times 10^{-18} \text{ m}^2,$$

and

$$\bar{c}_e = \sqrt{\frac{8 kT_e}{\pi m_e}} \approx 1.34 \times 10^6 \text{ m/s.}$$

$$\text{If also } n_e \approx 2.8 \times 10^{17} \text{ m}^{-3},$$

$$n_n \simeq 7.4 \times 10^{18} \text{ m}^{-3},$$

then

$$\left. \begin{aligned} v_{en} &= 7.4 \times 10^{18} \times 1.34 \times 10^6 \times 10^{-19} = 9.9 \times 10^5 \text{ s}^{-1} \\ v_{ei} &= 2.8 \times 10^{17} \times 1.34 \times 10^6 \times 4 \times 10^{-18} = 1.50 \times 10^6 \text{ s}^{-1} \end{aligned} \right\} v_e = 2.49 \times 10^6 \text{ s}^{-1}$$

At a point in the engine where $B = 100 \text{ gauss} = 0.01 \text{ Tesla}$,

$$\omega_c = \frac{1.6 \times 10^{-19} \times 10^{-2}}{0.91 \times 10^{-30}} = 1.76 \times 10^9 \text{ s}^{-1} \Rightarrow \beta = \frac{\omega_c}{v_e} = 706 \gg 1$$

Under these conditions, (8) reduces to

$$\boxed{D_{\perp} \simeq \frac{D_e}{\beta^2}} \quad (9)$$

or

$$\left[D_{\perp} = \frac{kT_e}{m_e} \frac{v_e}{v_e^2 + \omega_c^2} \simeq \frac{kT_e}{m_e} \frac{v_e}{\omega_c^2} \right] \quad (10)$$

This last form shows that collisions favor diffusion. In contrast, recall Equation (3), (no magnetic field, or $\beta \ll 1$), which shows that in that case, collisions impede

diffusion, Equation (10) also shows that D_{\perp} scales as $\frac{1}{B^2}$:

$$D_{\perp} \simeq \frac{kT_e}{m_e} \frac{v_e}{\frac{e^2 B^2}{m_e^2}} = \frac{m_e kT_e}{e^2} \frac{v_e}{B^2}$$

and so, increasing B should provide very strong confinement of electrons. With the given numbers, we find

$$D_e = \frac{kT_e}{m_e v_e} = \frac{1.38 \times 10^{-23} \times 46400}{0.91 \times 10^{-30} \times 2.49 \times 10^6} = 2.83 \times 10^5 \text{ m}^2/\text{s}$$

and

$$D_{\perp} = \frac{2.83 \times 10^5}{706^2} = 0.57 \text{ m}^2/\text{s}$$

A diffusing substance spreads (in 1-D) roughly as $x \sim 2\sqrt{Dt}$. So, to spread by 1 cm, electrons would require a time

$$t \sim \frac{x^2}{4D} = \frac{10^{-4} \text{ m}^2}{4 \times 0.57 \times 10^{-4} \text{ m}^2/\text{s}} = 4.5 \times 10^{-5} \text{ s}$$

It turns out, however, that electrons can diffuse faster than this in most cases. The physical reasons are apparently related to the "equivalent collisionality" produced by scattering of the electrons by small-scale plasma density fluctuations which are almost always present. This is the same situation that has kept tokamaks from delivering fusion power (only in that case it is the H⁺ ions that "leak through" the confining \bar{B} field).

Bohm obtained an empirical expression (with some theoretical guidance) for this so-called "anomalous diffusion"

$$D_{\text{Bohm}} = \frac{kT_e}{16eB} \quad (11)$$

and experiments in ion engines and Hall thrusters appear to confirm the 1/B dependence, but also seem to indicate a somewhat smaller diffusivity magnitude.

An often used expression is

$$D_{\text{anomalous}} = \frac{kT_e}{c_B eB} \quad (c_B \sim 16 - 100) \quad (12)$$

It is of some interest to see what collision frequency would produce the same diffusivity as these fluctuations:

$$\frac{kT_e}{c_B eB} \equiv \frac{m_e kT_e}{e^2} \frac{\nu_{\text{anomalous}}}{B^2} \Rightarrow \nu_{\text{anomalous}} = \frac{\omega_c}{c_B} \quad (13)$$

and so c_B can be thought of as the "Anomalous Hall Parameter". For modeling purposes, one often adds together $\nu_{\text{collision}} + \nu_{\text{anomalous}}$ in calculating diffusivity D_{\perp} by Equation (10).