

ERROR ANALYSIS (UNCERTAINTY ANALYSIS)

16.621 Experimental Projects Lab I

TOPICS TO BE COVERED

- Why do error analysis?
- If **we don't ever know the true value**, how do we estimate the error in the true value?
- Error propagation in the measurement chain
 - How do errors combine? (How do they behave in general?)
 - How do we do an **end-to-end uncertainty analysis**?
 - What are ways to mitigate errors?
- A hypothetical dilemma (probably nothing to do with anyone in the class)
 - When should I throw out some data that I don't like?
 - Answer: NEVER, but there are reasons to throw out data
- Backup slides: an example of an immense amount of money and effort directed at error analysis and mitigation - jet engine testing

ERROR AND UNCERTAINTY

- In engineering the word “error”, when used to describe an aspect of measurement does not necessarily carry the connotation of mistake or blunder (although it can!)
- **Error in a measurement means the inevitable uncertainty that attends all measurements**
- We cannot avoid errors in this sense
- We can ensure that they are as small as reasonably possible and that we have a reliable estimate of how small they are

[Adapted from Taylor, J. R, *An Introduction to Error Analysis; The Study of Uncertainties in Physical Measurements*]

USES OF UNCERTAINTY ANALYSIS (I)

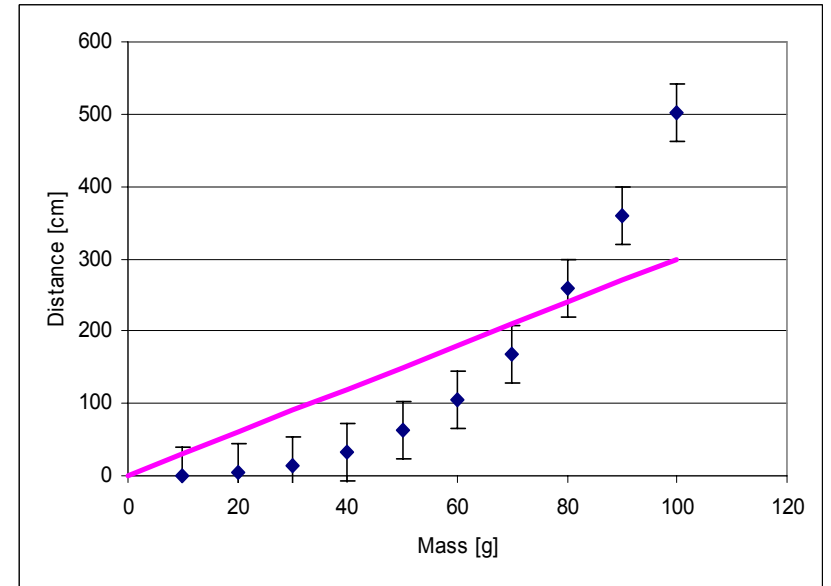
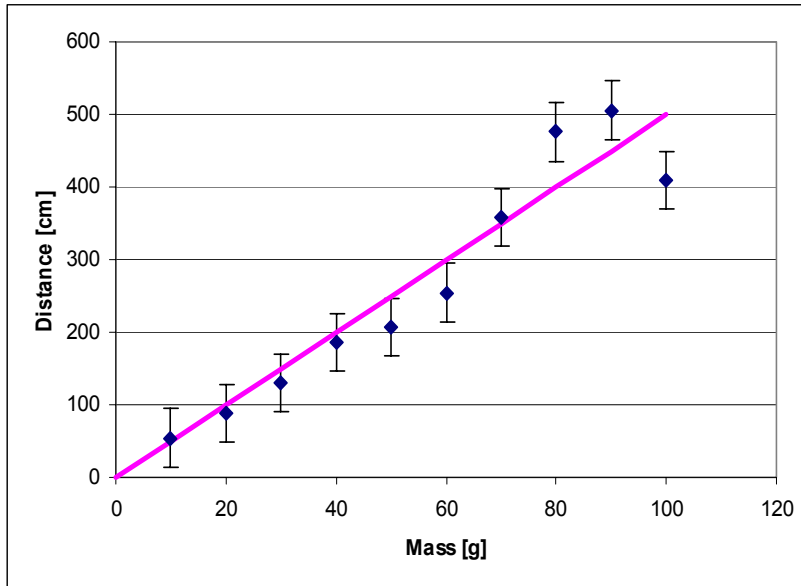
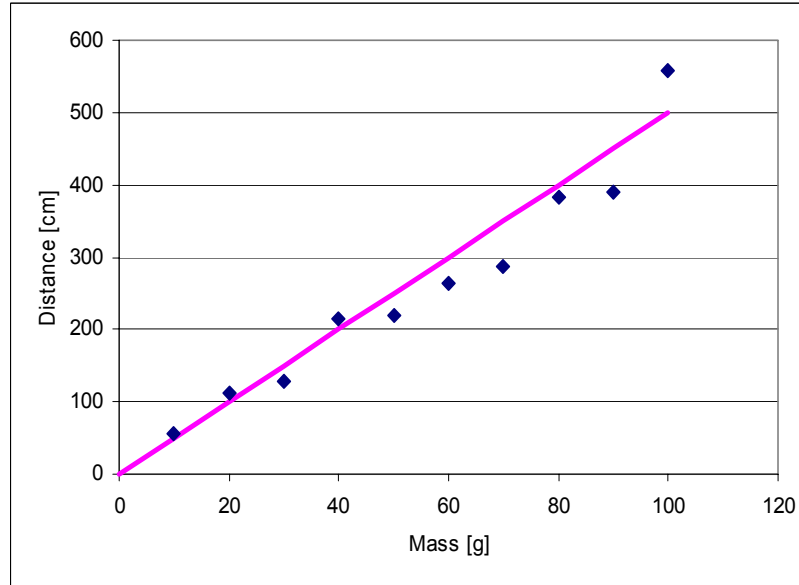
- **Assess experimental procedure including identification of potential difficulties**
 - **Definition of necessary steps**
 - **Gaps**
- **Advise what procedures need to be put in place for measurement**
- **Identify instruments and procedures that control accuracy and precision**
 - **Usually one, or at most a small number, out of the large set of possibilities**
- **Inform us when experiment cannot meet desired accuracy**

USES OF UNCERTAINTY ANALYSIS (II)

- **Provide the only known basis for deciding whether:**
 - **Data agrees with theory**
 - **Tests from different facilities (jet engine performance) agree**
 - **Hypothesis has been appropriately assessed (resolved)**
 - **Phenomena measured are real**
- **Provide basis for defining whether a closure check has been achieved**
 - **Is continuity satisfied (does the same amount of mass go in as goes out?)**
 - **Is energy conserved?**
- **Provide an integrated grasp of how to conduct the experiment**

[Adapted from Kline, S. J., 1985, "The Purposes of Uncertainty Analysis", *ASME J. Fluids Engineering*, pp. 153-160]

UNCERTAINTY ESTIMATES AND HYPOTHESIS ASSESSMENT



HOW DO WE DEAL WITH NOT KNOWING THE TRUE VALUE?

- In “all” real situations we don’t know the true value we are looking for
- We need to decide how to determine the best representation of this from our measurements
- We need to decide what the uncertainty is in our best representation

AN IMPLICATION OF NOT KNOWING THE TRUE VALUE

- **We easily divided errors into precision (bias) errors and random errors when we knew what the value was**
- **The target practice picture in the next slide is an example**
- **How about if we don't know the true value? Can we, by looking at the data in the slide after this, say that there are bias errors?**
- **How do we know if bias errors exist or not?**

RANDOM AND SYSTEMATIC ERRORS

RANDOM AND SYSTEMATIC ERRORS?

A TEAM EXERCISE

- **List the variables you need to determine in order to carry out your hypothesis assessment**
- **What uncertainties do you foresee? (Qualitative description)**
- **Are you more concerned about bias errors or random errors?**
- **What level of uncertainty in the final result do you need to assess your hypothesis in a rigorous manner?**
- **Can you make an estimate of the level of the uncertainty in the final result?**
 - **If so, what is it?**
 - **If not, what additional information do you need to do this?**

HOW DO WE COMBINE ERRORS?

- Suppose we measure quantity X with an error of dx and quantity Y with an error of dy
- What is the error in quantity Z if:
 - $Z = AX$ where A is a numerical constant such as π ?
 - $Z = X + Y$?
 - $Z = X - Y$?
 - $Z = XY$?
 - $Z = X/Y$?
 - Z is a general function of many quantities?

ERRORS IN THE FINAL QUANTITY

- $Z = X + Y$
- **Linear combination**
 - $Z + |dz| = X + |dx| + Y + |dy|$
 - **Error in Z is $|dz| = |dx| + |dy|$ BUT this is worst case**
- **For random errors we could have**
 - $|dz| = |dx| - |dy|$
or $|dy| - |dx|$
 - **These errors are much smaller**
- **In general if different errors are not correlated, are independent, the way to combine them is**

$$dz = \sqrt{dx^2 + dy^2}$$

- **This is true for random and bias errors**

THE CASE OF $Z = X - Y$

- Suppose $Z = X - Y$ is a number much smaller than X or Y

$$dz = \sqrt{dx^2 + dy^2}$$

- Say $\frac{dx}{X} = \frac{dy}{Y} = \varepsilon$ (say 2%)

- $\frac{dz}{Z} = \frac{\sqrt{2} dx}{X - Y}$ may be **much larger** than $\frac{dx}{X}$

- **MESSAGE ==> Avoid taking the difference of two numbers of comparable size**

ESTIMATES FOR THE TRUE VALUE AND THE ERROR

- **Is there a “best” estimate of the true value of a quantity?**
- **How do I find it?**
- **How do I estimate the random error?**
- **How do I estimate the bias error?**

SOME “RULES” FOR ESTIMATING RANDOM ERRORS AND TRUE VALUE

- An internal estimate can be given by repeat measurements
- Random error is generally of same size as standard deviation (root mean square deviation) of measurements
- **Mean of repeat measurements is best estimate of true value**
- Standard deviation of the mean (random error) is smaller than standard deviation of a single measurement by

$$\frac{1}{\sqrt{\text{Number of measurements}}}$$

- To increase precision by 10, you need 100 measurements

GENERAL RULE FOR COMBINATION OF ERRORS

- If $Z = F (X_1, X_2, X_3, X_4)$ is quantity we want
- The error in Z , dz , is given by our rule from before
- So, if the error F due to X_1 can be estimated as

$$dF_1 = \frac{\partial F}{\partial X_1} dx_1 \quad \text{and so on}$$

← Error in X_1

← Influence coeff.

$$dz = \sqrt{\left(\frac{\partial F}{\partial X_1}\right)^2 dx_1^2 + \left(\frac{\partial F}{\partial X_2}\right)^2 dx_2^2 + \dots + \left(\frac{\partial F}{\partial X_n}\right)^2 dx_n^2}$$

- The important consequence of this is that generally one or few of these factors is the main player and others can be ignored

DISTRIBUTION OF RANDOM ERRORS

- A measurement subject to many small random errors will be distributed “normally”
- Normal distribution is a Gaussian
- If x is a given measurement and X is the true value

$$\text{Gaussian or normal distribution} = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-X)^2}{2\sigma^2}}$$

- σ is the standard deviation

**DISTRIBUTION OF MEASUREMENTS SUBJECT TO
 N RANDOM ERRORS OF MAGNITUDE ε
(Solid Curves are Gaussian Distributions)**

A REVELATION

- The universal gas constant is

accepted $R = 8.31451 \pm 0.00007 \text{ J/mol K}$

- This is not a true value but can be “accepted” as one

ONE ADDITIONAL ASPECT OF COMBINING ERRORS

- **We have identified two different types of errors, bias (systematic) and random**
 - **Random errors can be assessed by repetition of measurements**
 - **Bias errors cannot; these need to be estimated using external information (mfrs. specs., your knowledge)**
- **How should the two types of errors be combined?**
 - **One practice is to treat each separately using our rule, and then report the two separately at the end**
 - **One other practice is to combine them as “errors”**
- **Either seems acceptable, as long as you show that you are going to deal (have dealt) with both**

REPORTING OF MEASUREMENTS

- **Experimental uncertainties should almost always be rounded to one significant figure**
- **The last significant figure in any stated answer should usually be of the same order of magnitude (in the same decimal position) as the uncertainty**

COMMENTS ON REJECTION OF DATA

- **Should you reject (delete) data?**
- **Sometimes one measurement appears to disagree greatly with all others. How do we decide:**
 - **Is this significant?**
 - **Is this a mistake?**
- **One criteria (Chauvenaut's criteria) is as follows**
 - **Suppose that errors are normally distributed**
 - **If measurement is more than M standard deviations (say 3), probability is < 0.003 that measurement should occur**
 - **Is this improbable enough to throw out measurement?**
- **The decision of “ridiculous improbability” [Taylor, 1997] is up to the investigator, but it allows the reader to understand the basis for the decision**
 - **If beyond this range, delete the data**

EVOLUTION OF A DATA SET (I)

[From *An Introduction to Experimentation* - E. Rabinowicz]

Stage 1: Original data of research worker

Stage 2: Data after research worker has "cleaned up the data" and removed a few points

EVOLUTION OF A DATA SET (II)

Stage 3: Data after supervisor has removed a few more points

Stage 4: Data after director of research has removed a few more points still. This figure is published

A CAVEAT ON REJECTION OF DATA

- **If more than one measurement is different, it may be that something is really happening that has not been envisioned, e.g., discovery of radon**
- **You may not be controlling all the variables that you need to**
- **Bottom line: Rigorous uncertainty analysis can give rationale to decide what data to pay attention to**

SUMMARY

- **Both the number and the fidelity of the number are important in a measurement**
- **We considered two types of uncertainties, bias (or systematic errors) and random errors**
- **Uncertainty analysis addresses fidelity and is used in different phases of an experiment, from initial planning to final reporting**
 - **Attention is needed to ensure uncertainties do not invalidate your efforts**
- **In propagating uncorrelated errors from individual measurement to final result, use the square root of the sums of the squares of the errors**
 - **There are generally only a few main contributors (sometimes one) to the overall uncertainty which need to be addressed**
- **Uncertainty analysis is a critical part of “real world” engineering projects**

SOME REFERENCES I HAVE FOUND USEFUL

- **Baird, D. C., 1962, *Experimentation: An Introduction to Measurement Theory and Experiment*, Prentice-Hall, Englewood Cliffs, NJ**
- **Bevington, P. R, and Robinson, D. K., 1992, *Data Reduction and Error Analysis for the Physical Sciences*, McGraw-Hill, New York, NY**
- **Lyons, L., 1991, *A Practical Guide to Data Analysis for Physical Science Students*, Cambridge University Press, Cambridge, UK**
- **Rabinowicz, E, 1970, *An Introduction to Experimentation*, Addison-Wesley, Reading, MA**
- **Taylor, J. R., 1997, *An Introduction to Error Analysis*, University Science Books, Sausalito, CA**

BACKUP EXAMPLE: MEASUREMENT OF JET ENGINE PERFORMANCE

- **We want to measure Thrust, Airflow, and Thrust Specific Fuel Consumption (TSFC)**
 - **Engine program can be \$1B or more, take three years or more**
 - **Engine companies give guarantees in terms of fuel burn**
 - **Engine thrust needs to be correct or aircraft can't take off in the required length**
 - **Airflow fundamental in diagnosing engine performance**
 - **These are basic and essential measures**
- **How do we measure thrust?**
- **How do we measure airflow?**
- **How do we measure fuel flow?**

DIRECT-CONNECT ENGINE INSTALLATION

[from Covert, E.E., ed., *Thrust and Drag:
Its Prediction and Verification*]

ALTITUDE TEST CELL WITH OVERHEAD THRUST MEASURING SYSTEMS

[from Covert, E.E., ed., *Thrust and Drag:
Its Prediction and Verification*]

THRUST STANDS

- **In practice, thrust is measured with load cells**
- **The engines, however, are often part of a complex test facility and are connected to upstream ducting**
- **There are thus certain systematic errors which need to be accounted for**
- **The level of uncertainty in the answer is desired to be less than one per cent**
- **There are a lot of corrections to be made to the raw data (measured load) to give the thrust**

THRUST MEASUREMENT ERROR CHARTS

– Afterburning Turbojet, Military Power, 40 K, M = 1.2 –

[from Covert, E.E., ed., *Thrust and Drag:
Its Prediction and Verification*]

FUEL FLOW ERROR CHART

[from Covert, E.E., ed., *Thrust and Drag:
Its Prediction and Verification*]

TEST STAND-TO-TEST STAND DIFFERENCES

- **Want to have a consistent view of engine performance no matter who quotes the numbers**
- **This means that different test stands must be compared to see the differences**
- **Again, this is a major exercise involving the running of a jet engine in different locations under specified conditions**
- **The next slide shows the level of differences in the measurements**

INTERFACILITY DATA COMPARISON

[from Covert, E.E., ed., *Thrust and Drag:
Its Prediction and Verification*]