

- 2) (10p) We will solve the KdV equation numerically using the method of lines and finite difference approximations for the space derivatives. Rewrite the equation as

$$\frac{\partial u}{\partial t} = 6uu_x - u_{xxx}, \quad (3)$$

and derive a second-order accurate difference approximation of the right-hand side.

- 3) (15p) For the time integration, we will use a fourth order Runge-Kutta scheme:

$$\alpha^1 = \Delta t f(u^i) \quad (4)$$

$$\alpha^2 = \Delta t f(u^i + \alpha^1/2) \quad (5)$$

$$\alpha^3 = \Delta t f(u^i + \alpha^2/2) \quad (6)$$

$$\alpha^4 = \Delta t f(u^i + \alpha^3) \quad (7)$$

$$u^{i+1} = u^i + \frac{1}{6}(\alpha^1 + 2\alpha^2 + 2\alpha^3 + \alpha^4). \quad (8)$$

The stability region for this scheme consists of all z such that $|1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24}| \leq 1$. In particular, all points on the imaginary axis between $\pm i2\sqrt{2}$ are included.

Our equation (1) is non-linear, and to make a stability analysis we first have to linearize it. In this case, it turns out that the stability will be determined by the discretization of the third-derivative term u_{xxx} . Therefore, consider the simplified problem

$$\frac{\partial u}{\partial t} = -u_{xxx}, \quad (9)$$

and use von Neumann stability analysis to derive an expression for the maximum allowable time-step Δt in terms of Δx .

- 4) (20p) Write a program that solves the equation using your discretization. Solve it in the region $-8 \leq x \leq 8$ with a grid size $\Delta x = 0.1$, and use periodic boundary conditions:

$$x(-8) = x(8). \quad (10)$$

Integrate from $t = 0$ to $t = 2$, using an appropriate time-step that satisfies the condition you derived above. For each of the initial conditions below, plot the solution at $t = 2$ and comment on the results.

- a. To begin with, use a single soliton (2) as initial condition, that is, $u(x, 0) = u_1(x, 0)$. Set $v = 16$ and $x_0 = 0$.
- b. The one-soliton solution looks almost like a Gaussian. Try $u(x, 0) = -8e^{-x^2}$.
- c. Try the two-soliton solution $u(x, 0) = -6/\cosh^2(x)$.
- d. Create “your own” two-soliton solution by superposing two one-soliton solutions with $v = 16$ and $v = 4$ (both with $x_0 = 0$).
- e. Same as before, but with $v = 16, x_0 = 4$ and $v = 4, x_0 = -4$. Describe what happens when the two solitons cross (amplitudes, velocities), and after they have crossed.

Problem 2 - Traffic Flow (50p)

Problem Statement

Consider the traffic flow problem, described by the non-linear hyperbolic equation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} = 0 \quad (11)$$

with $\rho = \rho(x, t)$ the density of cars (vehicles/km), and $u = u(x, t)$ the velocity. Assume that the velocity u is given as a function of ρ :

$$u = u_{\max} \left(1 - \frac{\rho}{\rho_{\max}} \right). \quad (12)$$

With u_{\max} the maximum speed and $0 \leq \rho \leq \rho_{\max}$. The flux of cars is therefore given by:

$$f(\rho) = \rho u_{\max} \left(1 - \frac{\rho}{\rho_{\max}} \right). \quad (13)$$

We will solve this problem using a first order finite volume scheme:

$$\rho_i^{n+1} = \rho_i^n - \frac{\Delta t}{\Delta x} \left(F_{i+\frac{1}{2}}^n - F_{i-\frac{1}{2}}^n \right). \quad (14)$$

For the numerical flux function, we will consider two different schemes:

a) Roe's Scheme

The expression of the numerical flux is given by:

$$F_{i+\frac{1}{2}}^R = \frac{1}{2} [f(\rho_i) + f(\rho_{i+1})] - \frac{1}{2} |a_{i+\frac{1}{2}}| (\rho_{i+1} - \rho_i) \quad (15)$$

with

$$a_{i+\frac{1}{2}} = u_{\max} \left(1 - \frac{\rho_i + \rho_{i+1}}{\rho_{\max}} \right). \quad (16)$$

Note that $a_{i+\frac{1}{2}}$ satisfies

$$f(\rho_{i+1}) - f(\rho_i) = a_{i+\frac{1}{2}} (\rho_{i+1} - \rho_i). \quad (17)$$

b) Godunov's Scheme

In this case the numerical flux is given by:

$$F_{i+\frac{1}{2}}^G = f \left(\rho \left(x_{i+\frac{1}{2}}, t^{n+} \right) \right) = \begin{cases} \min_{\rho \in [\rho_i, \rho_{i+1}]} f(\rho), & \rho_i < \rho_{i+1} \\ \max_{\rho \in [\rho_i, \rho_{i+1}]} f(\rho), & \rho_i > \rho_{i+1}. \end{cases} \quad (18)$$

Questions

- 1) (25p) For both Roe's Scheme and Godunov's Scheme, look at the problem of a traffic light turning green at time $t = 0$. We are interested in the solution at $t = 2$ using both schemes. What do you observe for each of the schemes? Explain briefly why the behavior you get arises.

Use the following problem parameters:

$$\begin{aligned}\rho_{\max} &= 1.0, & \rho_L &= 0.8 \\ u_{\max} &= 1.0 \\ \Delta x &= \frac{4}{400}, & \Delta t &= \frac{0.8\Delta x}{u_{\max}}\end{aligned}\tag{19}$$

The initial condition at the instant when the traffic light turns green is

$$\rho(0) = \begin{cases} \rho_L, & x < 0 \\ 0, & x \geq 0 \end{cases}\tag{20}$$

For the rest of this problem use only the scheme(s) which are valid models of the problem.

- 2) (25p) Simulate the effect of a traffic light at $x = -\frac{\Delta x}{2}$ which has a period of $T = T_1 + T_2 = 2$ units. Assume that the traffic light is $T_1 = 1$ units on red and $T_2 = 1$ units on green. Assume a sufficiently high flow density of cars (e.g. set $\rho = \frac{\rho_{\max}}{2}$ on the left boundary – giving a maximum flux), and determine the average flow, or capacity of cars over a time period T .

The average flow can be approximated as

$$\dot{q} = \frac{1}{N_T} \sum_{n=1}^{N_T} f^n = \frac{1}{N_T} \sum_{n=1}^{N_T} \rho^n u^n,\tag{21}$$

where N_T is the number of time steps for each period T . You should run your computation until \dot{q} over a time period does not change. Note that by continuity \dot{q} can be evaluated over any point in the interior of the domain (in order to avoid boundary condition effects, we consider only those points on the interior domain).

Note: A red traffic light can be modeled by simply setting $F_{i+\frac{1}{2}} = 0$ at the position where the traffic light is located.

- 3) **BONUS:** (Possible +10p¹) Assume now that we simulate two traffic lights, one located at $x = 0$, and the other at $x = 0.15$, both with a period T . Calculate the road capacity (= average flow) for different delay factors. That is if the first light turns green at time t , then the second light will turn green at $t + \tau$. Solve for $\tau = k\frac{T}{10}$, $k = 0, \dots, 9$. Plot your results of capacity vs τ and determine the optimal delay τ .

¹Only applied to gain a maximum of 100%. Additional bonus points are not carried over to future assignments.