1. a) i) The ratio is proportional to the Rutherford scattering rate and ring area.

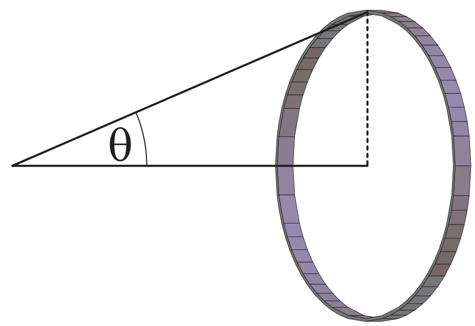

Figure I: Ring area =p $2\pi\sin\theta d\theta$

$$\Rightarrow \frac{R_{2\pi/4}}{R_{\pi/4}} = \frac{\sin\theta}{\sin^4\frac{\theta}{2}}\frac{\sin^4\frac{\theta'}{2}}{\sin\theta'} = \left(\frac{0.383}{0.924}\right)^4 = 2.95\times10^{-2} \quad (1\text{-}1)$$

ii) The differential cross section $\frac{d\sigma}{d\Omega}(\theta,\phi)$ is that ratio of the scattering rate per solid angle $\frac{dR_1}{d\Omega}$, to the incident intensity I.

$$\frac{d\sigma}{d\Omega} = \frac{dR_1/d\Omega}{I}\ [\mathrm{m}^2/\mathrm{sterad}] \quad (1\text{-}2)$$

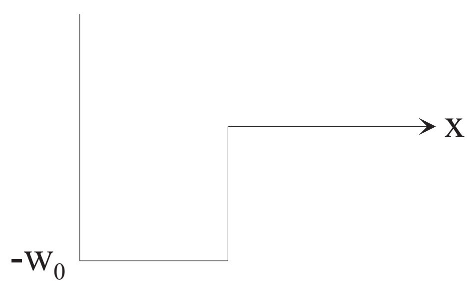

Figure II: $W_{\mathrm{KIN}} = h\nu - W_0$

b) i) Below ν_0 the photons do not have sufficient energy to release the bound electron from the surface.

ii) Slope is:

$$h = \left(4.31\times10^{-15}\ \frac{\mathrm{eV}}{\mathrm{Hz}}\right)\cdot\left(1.602\times10^{-19}\ \frac{\mathrm{J}}{\mathrm{eV}}\right) = 6.9\times10^{-34}\ \mathrm{J{\cdot}s} \quad (1\text{-}3)$$

Work function **from this measurement**:

$$W_0 = h\nu_0 = (6.9\times10^{-34}\ \mathrm{J{\cdot}s})\cdot(1.12\times10^{15}\ \mathrm{Hz}) = 7.7\times10^{-19}\ \mathrm{J} \quad (1\text{-}4)$$

c) The Compton shift formula is derived from conservation of energy and momentum and so still applicable here:

$$\lambda' - \lambda = \frac{h}{m_p c}(1 - \cos\theta), \tag{1-5}$$

where m_p is the proton mass.
$\therefore$ maximum shift for $\theta = \pi$,

$$\Delta\lambda_{\max} = \lambda' - \lambda = \frac{2h}{m_p c} \tag{1-6}$$

$$\frac{\Delta\lambda}{\lambda} = \frac{2h}{m_p c}\frac{1}{\lambda} = \frac{2\cdot\left(6.6\times10^{-34}\ \text{J}\cdot\text{s}\right)}{\left(1.67\times10^{-27}\ \text{kg}\right)\cdot\left(3\times10^{8}\ \frac{\text{m}}{\text{s}}\right)}\cdot\frac{1}{\left(10^{-10}\ \text{m}\right)} \tag{1-7}$$

Fractional frequency shift is $\frac{\Delta\nu}{\nu}$,

$$\lambda = \frac{\text{v}}{\nu} \Rightarrow |d\lambda| = \frac{\text{v}}{\nu^2}|d\nu|. \tag{1-8}$$

$$\Rightarrow \frac{\Delta\nu}{\nu} = \frac{\Delta\lambda}{\lambda} = 2.6\times10^{-5} \qquad (\Delta\lambda \ll \lambda) \tag{1-9}$$

2. a) $\int dx|\psi|^2 = w|C|^2 + w|C|^2 = 2w|C|^2 = 1$

$$\Rightarrow \boxed{C = \frac{1}{\sqrt{2w}}}$$

b)

$$\langle x\rangle = \int_{-\infty}^{\infty} dx x|\psi(x)|^2 = 0 \qquad \text{(odd integrand)} \tag{1-10}$$

$$\begin{aligned}
\langle x^2\rangle &= \int_{-\infty}^{\infty} dx x^2|\psi(x)|^2 = |A|^2\int_{-\frac{d}{2}-\frac{w}{2}}^{-\frac{d}{2}+\frac{w}{2}} dx x^2 + |A|^2\int_{\frac{d}{2}-\frac{w}{2}}^{\frac{d}{2}+\frac{w}{2}} dx x^2 \\
&= |A|^2\left\{\frac{1}{3}x^3\Big|_{-\frac{d}{2}-\frac{w}{2}}^{-\frac{d}{2}+\frac{w}{2}} + \frac{1}{3}x^3\Big|_{\frac{d}{2}-\frac{w}{2}}^{\frac{d}{2}+\frac{w}{2}}\right\} \\
&= \frac{1}{2w}\left\{-\frac{1}{3}\left(\frac{d}{2}-\frac{w}{2}\right)^3 + \frac{1}{3}\left(\frac{d}{2}+\frac{w}{2}\right)^3 + \frac{1}{3}\left(\frac{d}{2}+\frac{w}{2}\right)^3 - \frac{1}{3}\left(\frac{d}{2}-\frac{w}{2}\right)^3\right\} \\
&\approx \frac{1}{48w}\left\{3d^2w + 3d^2w + 3d^2w + 3d^2w\right\} = \frac{1}{4}d^2 \qquad (\text{in limit } w \ll d)
\end{aligned}$$

or simply,

$$\langle x^2\rangle = \frac{1}{2}\left(-\frac{d}{2}\right)^2 + \frac{1}{2}\left(\frac{d}{2}\right)^2 = \frac{1}{4}d^2 \tag{1-11}$$

c)

$$\begin{aligned}
\phi(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \psi(x) e^{-ikx} \\
&= \frac{c}{\sqrt{2\pi}} \cdot \left(\int_{\frac{-d-w}{2}}^{\frac{-d+w}{2}} dx e^{-ikx} + \int_{\frac{d-w}{2}}^{\frac{d+w}{2}} dx e^{-ikx} \right) \\
&= \frac{c}{-ik\sqrt{2\pi}} \cdot \left(e^{-ikx} \Big|_{\frac{-d-w}{2}}^{\frac{-d+w}{2}} + e^{-ikx} \Big|_{\frac{d-w}{2}}^{\frac{d+w}{2}} \right) \\
&= \frac{-c}{ik\sqrt{2\pi}} \cdot \left(e^{-ik\left(\frac{-d+w}{2}\right)} - e^{-ik\left(\frac{-d-w}{2}\right)} + e^{-ik\left(\frac{d+w}{2}\right)} - e^{-ik\left(\frac{d-w}{2}\right)} \right) \\
&= \frac{c}{ik\sqrt{2\pi}} \cdot \left(e^{ik\left(\frac{d+w}{2}\right)} - e^{-ik\left(\frac{d+w}{2}\right)} - e^{ik\left(\frac{d-w}{2}\right)} + e^{-ik\left(\frac{d-w}{2}\right)} \right) \\
&= \frac{2c}{k\sqrt{2\pi}} \cdot \left(\sin k\frac{d+w}{2} - \sin k\frac{d-w}{2} \right) \\
&= \frac{1}{\sqrt{\pi}} \cdot \left(\frac{\sin k\frac{d+w}{2} - \sin k\frac{d-w}{2}}{k\sqrt{w}} \right) \qquad (1\text{-}12)
\end{aligned}$$

Probability density for p is $\frac{1}{\hbar}\left|\phi\left(k = \frac{p}{\hbar}\right)\right|^2$ i.e.,

$$|\tilde{\phi}(p)|^2 = \frac{\left(\sin p\frac{d+w}{2\hbar} - \sin p\frac{d-w}{2\hbar}\right)^2}{\pi p^2 w/\hbar} = \frac{\hbar}{\pi p^2 w}\left(\sin\frac{p(d+w)}{2\hbar} - \sin\frac{p(d-w)}{2\hbar}\right) \qquad (1\text{-}13)$$

for momentum or

$$|\phi(k)|^2 = \frac{\left(\sin\frac{k(d+w)}{2} - \sin\frac{k(d-w)}{2}\right)^2}{\pi k^2 w} \qquad (1\text{-}14)$$

d) The phase shift causes the interference pattern to shift in position, but the contrast is unchanged.

3. a) The Heisenberg uncertainty principle gives

$$\Delta x \Delta p \leq \frac{\hbar}{2}, \qquad (1\text{-}15)$$

so to resolve proton requires

$$\Delta p = \frac{\hbar}{2R} = \frac{1.05 \times 10^{-34}\ \text{J}\cdot\text{s}}{2 \times 10^{-15}\ \text{m}} = 5.3 \times 10^{-20}\ \text{kg}\cdot\frac{\text{m}}{\text{s}}, \qquad (1\text{-}16)$$

$$m_e c = \left(9.1 \times 10^{-31}\ \text{kg}\right)\cdot\left(3 \times 10^{8}\ \frac{\text{m}}{\text{s}}\right) = 2.7 \times 10^{-22}\ \text{kg}\cdot\frac{\text{m}}{\text{s}}. \qquad (1\text{-}17)$$

$$\Rightarrow \Delta p \gg m_e c \Rightarrow \text{relativistic situation}$$

b) For relativistic situation, $E = pc$

$$E = \left(5.3 \times 10^{-20}\ \text{kg}\cdot \frac{\text{m}}{\text{s}}\right) \cdot \left(3 \times 10^{8}\ \frac{\text{m}}{\text{s}}\right) = 1.6 \times 10^{-11}\ \text{J}. \tag{1-18}$$

c) For a proton,

$$m_p c = \left(1.67 \times 10^{-27}\ \text{kg}\right) \cdot \left(3 \times 10^{8}\ \frac{\text{m}}{\text{s}}\right) = 5 \times 10^{-19} \text{kg} \cdot \frac{\text{m}}{\text{s}}. \tag{1-19}$$

$\Rightarrow$ non-relativistic

$$\therefore E = \frac{(\Delta p)^2}{2m_p} = 8.4 \times 10^{-34}\ \text{J} \tag{1-20}$$

i.e., much less particle energy needed than for electron. Based on this, one would rather build a proton-proton collider.

4. a) If particle is localized, i.e. Δx is small, then the Heisenberg uncertainty relation implies $\Delta p = \frac{\hbar}{2\Delta x}$ is large and so kinetic energy is large.

b) Total energy

$$E = \frac{(\Delta p)^2}{2m} + \frac{1}{2} m\omega^2 (\Delta x)^2 = \frac{\hbar^2}{8m(\Delta x)^2} + \frac{1}{2} m\omega^2 (\Delta x)^2 \tag{1-21}$$

Minimizing:

$$\frac{\partial E}{\partial x} = 0 = -\frac{2\hbar^2}{8m(\Delta x)^3} + m\omega^2 \Delta x \tag{1-22}$$

$$\Rightarrow \frac{\hbar^2}{4m} = mw^2 (\Delta x)^4 \Rightarrow (\Delta x)^4 = \frac{\hbar^2}{4m^2\omega^2} \tag{1-23}$$

$$\Rightarrow \Delta x = \left(\frac{\hbar}{2m\omega}\right)^{\frac{1}{2}} \tag{1-24}$$