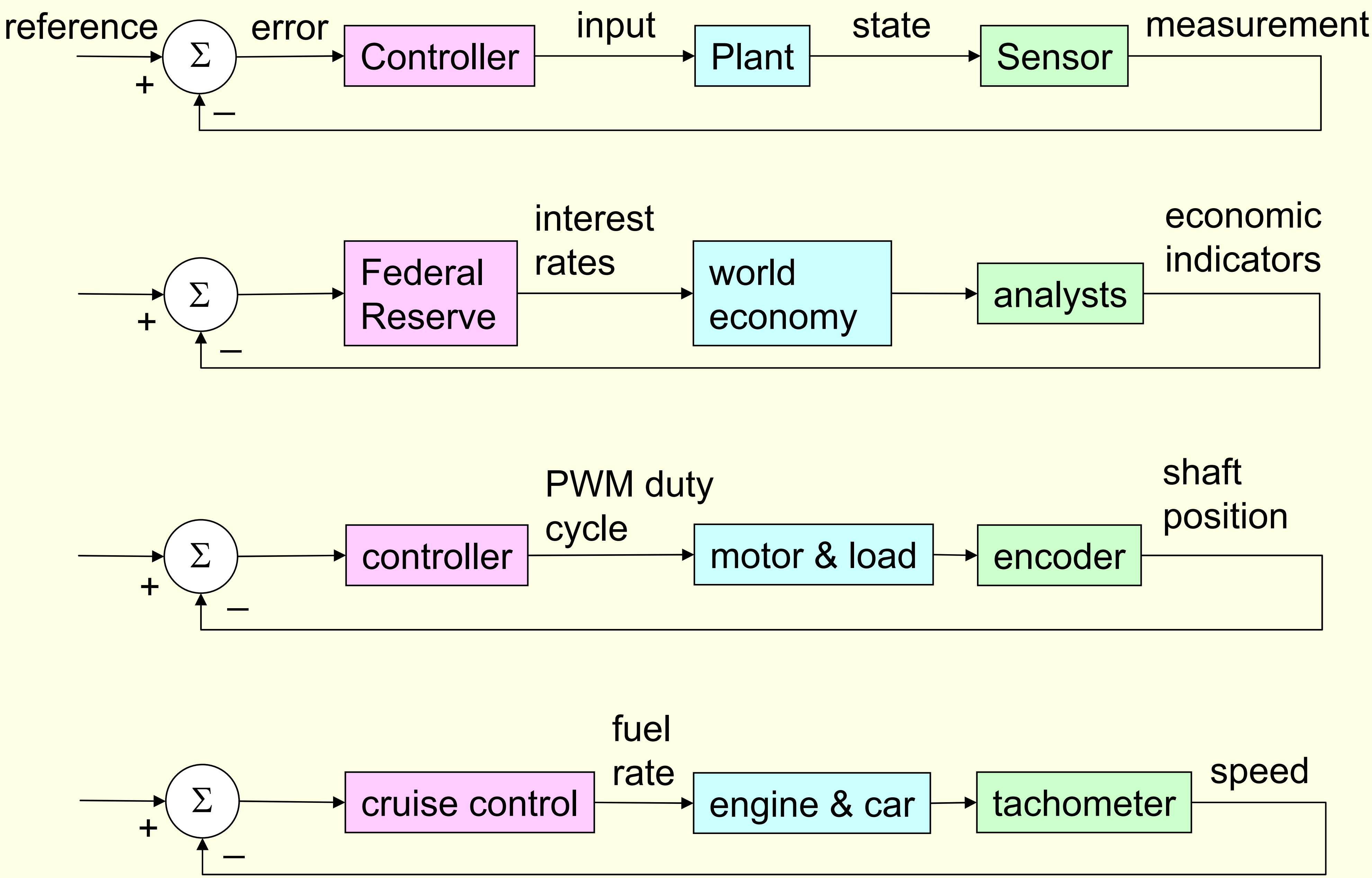
# Feedback Control

# Components of Engineered Feedback Systems

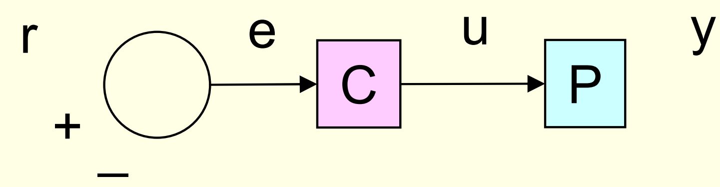
- **Plant**: the system whose behavior is to be controlled.
  *Examples: vehicle attitude, temperature, chemical process, business accounting, team and personal relationships, global climate*
- **Actuator**: systems which alter the behavior of the plant.
  *Examples: motor, heater, valve, law enforcement (!), pump, FET, hydraulic ram, generator, US Mint*
- **Sensor**: system which measures certain states of the plant.
  *Examples: thermometer, voltmeter, Geiger counter, opinion poll, balance sheet, financial analyst*
- **Controller**: translates sensor output into actuator input.
  *Examples: computer, analog device, human interface, committee*

- Extreme variability in time scales:
  - active noise cancellation requires ~100 *kiloHertz* sensing and actuation
  - Social Security is assessed and corrected at ~3 *nanoHertz* (10 years)

*Feedback fundamentally creates a new dynamics!*



# Basics in the Frequency Domain



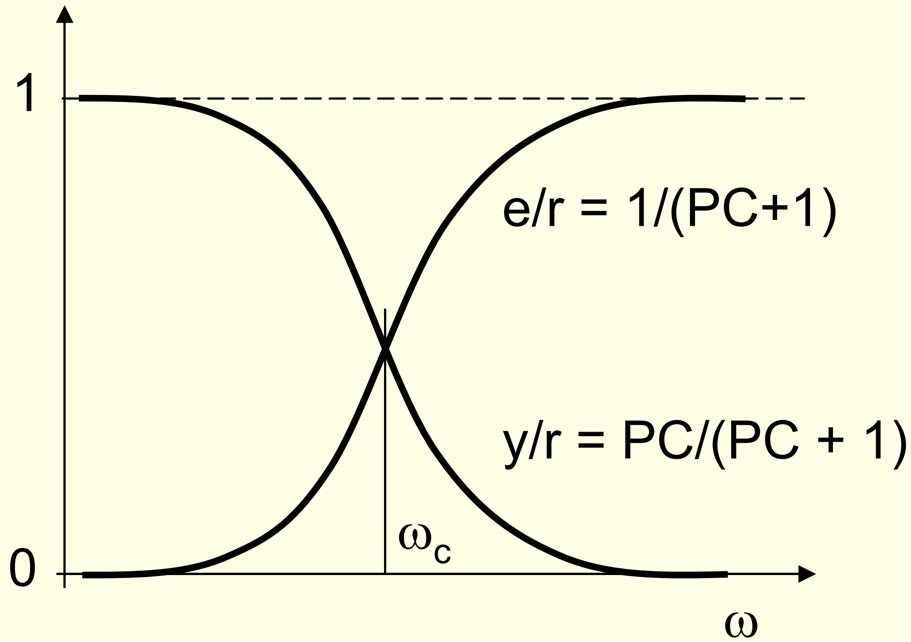

e = r – y
u = Ce = C(r-y)
y = Pu = PCe = PC(r-y) → (PC + 1)y = PCr → **y / r = PC / (PC + 1)**

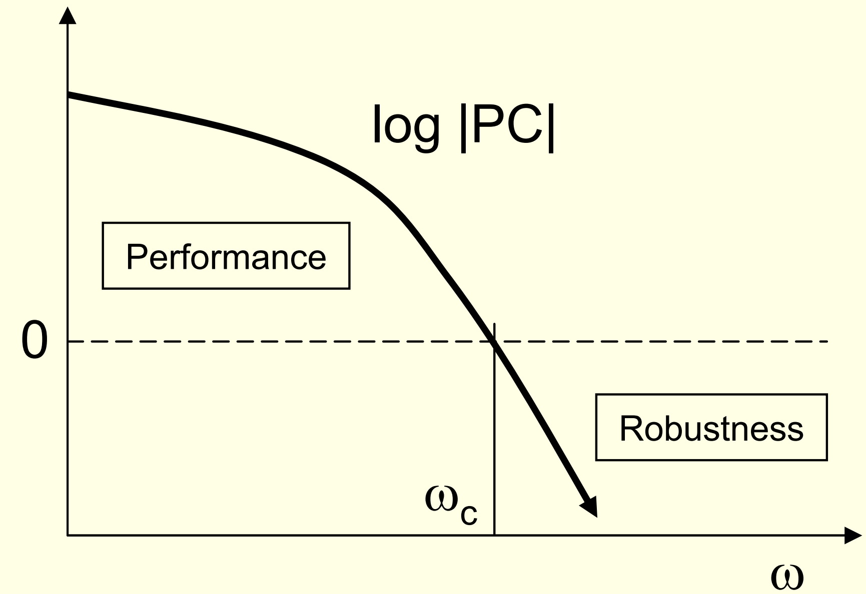
Similarly, e = r – y = r – PCe → (PC+1)e = r → **e / r = 1 / (PC + 1)**
u = C(r-Pu) → (PC + 1)u = Cr → **u / r = C / (PC + 1)**

*Why can we do this? Convolution in time domain = Multiplication in freq. domain!*

*P must roll off at high frequencies – because no physical plant can respond to input at arbitrarily high frequency. Same with controller C.*

- Ideal case: e is a small fraction of r: e/r << 1, equivalent to y/r ~ 1
- This implies mag (PC + 1) >> 1 or mag (PC) >> 1.
- If plant P is given, then C has to be *designed* to make PC big.
- But mag (u / r) ~ mag(1 / P): HUGE when P gets small at high frequencies→ excessive control action which will saturate or break actuators, excite unmodelled plant behavior, etc.. ← issues of *robustness*




Good tracking only possible at low frequencies → leads to a "formula" for design:

Make |PC| *large at low frequencies*, e/r ~ 0, y/r ~ 1;
Good regulation and tracking at low frequencies

Make |PC| *small at high frequencies*, e/r ~ 1, y/r ~ 0, u/r ~ C
Poor tracking at high frequencies, but reasonable control action

The frequency where |PC| = 1 is the <u>crossover frequency</u> $\omega_c$ ;
Above this point, closed loop t.f. y/r = PC/(PC+1) drops off to zero.
So $\omega_c$ is about the *bandwidth* of the closed-loop t.f.

# LaPlace vs. Fourier XFM

Fourier Transform integrates **$x(t)\ e^{-j\omega t}$** over the time range from <u>negative infinity</u> to positive infinity

Laplace Transform integrates **$x(t)\ e^{-st}$** over the time range from <u>zero</u> to positive infinity

Result: $X(j\omega)$ can describe *acausal* systems, $X(s)$ describes only *causal* ones!

Many important results of Fourier Transform carry over to LaPlace Transform:

$\mathcal{L}(x(t)) = X(s)$ (notation)

$\mathcal{L}(ax(t)) = a\,X(s)$ (linearity)

$\mathcal{L}(x(t) * y(t)) = X(s)Y(s)$ (convolution)

$\mathcal{L}(x_t(t)) \leftrightarrow sX(s)$ (first time derivative)

$\mathcal{L}(x_{tt}(t)) \leftrightarrow s^2X(s)$ (second and higher time derivatives)

$\mathcal{L}(\int x(t)dt) \leftrightarrow X(s) / s$ (time integral)

$\mathcal{L}(\delta(t)) = 1$ (unit impulse)

$\mathcal{L}(1(t)) = 1/s$ (unit step)

# LaPlace Transform and Stability

- For linear systems, stability of a system refers to whether the impulse response has *exponentially growing components*.
- *No pre-determined input can stabilize an unstable system; no pre-determined input can destabilize a stable system.*
- Some examples you can work out:

$$\mathcal{L}(e^{-\alpha t}) = 1 / (s + \alpha)$$

$$\mathcal{L}(t\, e^{-\alpha t}) = 1 / (s + \alpha)^2$$

$$\mathcal{L}[\, e^{-\alpha t} \sin(\omega t)\, ] = \omega / (s^2 + 2\alpha s + \alpha^2 + \omega^2)$$

$$\mathcal{L}[\, \omega_d\, e^{-\zeta \omega n t} \sin(\omega_d t) / (1-\zeta^2)\, ] = \omega_n^2 / (s^2 + 2\zeta\omega_n s + \omega_n^2)$$

**Major observation: stable signal ←→ roots of $\mathcal{L}$ denominator have negative real parts: TRUE FOR ALL FIRST- AND SECOND-ORDER SYSTEMS**

# Decoding the transfer function

Numerator polynomials are a snap:

$(s + 2)/(s^2+s+5) = s/(s^2 + s + 5) + 2/(s^2+s+5)$

"input derivative plus two times the input, divided by the denominator"

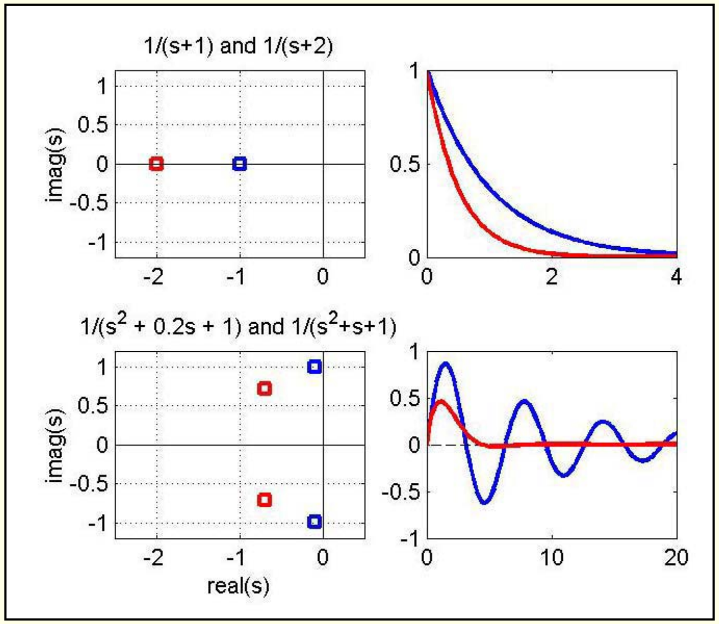
For higher-order polynomials in the denominator: use partial fractions, e.g.,

$(s+1)/(s+2)(s+3)(s+4) = -0.5/(s+2) + 2/(s+3) -1.5/(s+4)$ *(all real poles)*

$(s+1)/s(s^2+s+1) = -s/(s^2+s+1) + 1/s$ *(some complex poles)*

Any high-order transfer function can always be broken down into a sum of transfer functions with factored first- and second-order polynomials in the denominator.

**stability ←→ the roots of the characteristic equation have negative real part.**




More details:

*real negative root* $-\alpha$: the mode decays with time constant $1/\alpha$

*complex roots at* $-\omega_n\zeta$ +/- $j\omega_d$: the mode decays with frequency $\omega_d$ and exponential envelope having time constant $1/\zeta\omega_n$

# Example with a double integrator: e.g., a motor or dynamic positioning

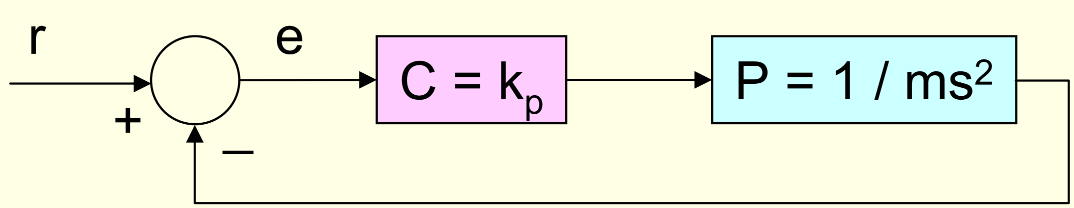
System is $mx_{tt}(t) = u(t)$ where:

$m$ is mass
$x_{tt}(t)$ is double time derivative of position
$u(t)$ is control action; thrust

Let a *Control law* be: **$u = -k_p x$ (Proportional Control: P)**
*Closed-loop* system dynamics become $mx_{tt} + k_p x = 0$
*Response to an initial condition* is undamped oscillations at frequency $\omega_n = \mathrm{sqrt}(k_p/m)$

$P = 1/ms^2$
$C = k_p$
$PC = k_p/ms^2 \rightarrow$

$e/r = 1/(PC + 1)$
$= ms^2 / (ms^2 + k_p)$




Tracking error is small when s is small; large when s is large, as desired.
BUT characteristic equation $ms^2 + k_p = 0$ has two imaginary poles – undamped!

Try the control law $u = -k_p x - k_d x_t$ **(Proportional + Derivative: PD)**

Closed-loop system dynamics become $m x_{tt} + k_d x_t + k_p x = 0$

Recall for a second-order underdamped oscillator:

$0 < k_d < 2\ \mathrm{sqrt}(k_p/m)$

$\omega_n = \mathrm{sqrt}(k_p/m)$ (undamped natural frequency)

$\zeta = k_d / 2\ \mathrm{sqrt}(k_p m)$ (damping ratio)

$\omega_d = \omega_n\ \mathrm{sqrt}(1-\zeta^2)$ (damped natural frequency)

Response to an initial condition is either:

- *Damped oscillations at frequency* $\omega_d = \mathrm{sqrt}(1-\zeta^2)\omega_n$, *inside an exponential envelope with time constant* $1/\zeta\omega_n$ OR
- *Sum of two decaying exponentials* (overdamped case)

Consider a constant disturbance: $mx_{tt} + k_d x_t + k_p x = F$;

System will settle at $x = F/k_p$; this is a steady-state error!

But $k_p$ cannot be increased arbitrarily – natural frequency will be too high and too much control action

Try the control law $u = -k_p x - k_d x_t - k_i \int x \, dt$

**(Proportional + Derivative + Integral: PID)**

Closed-loop system dynamics become

$$mx_{tt} + k_d x_t + k_p x + ki \int x \, dt = F$$

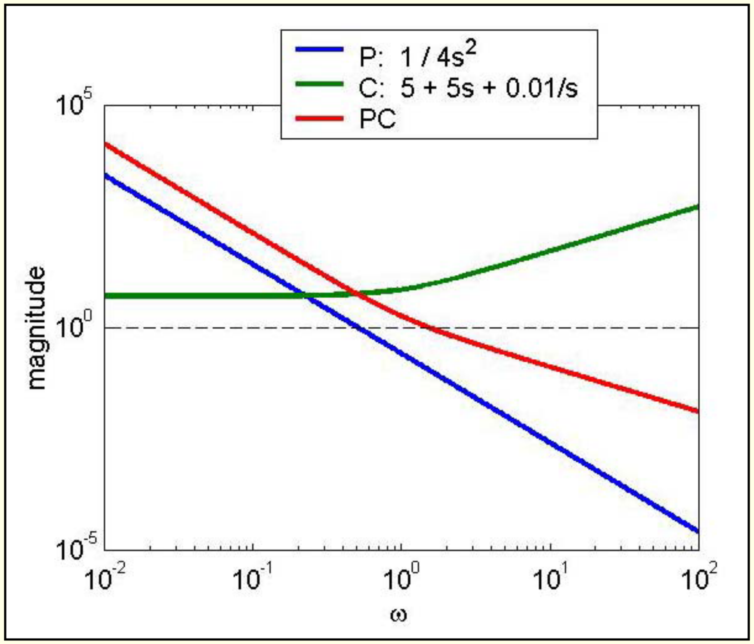
If the system is stable ($ms^3 + k_d s^2 + k_p s + k_i = 0$ has roots with negative real part), then differentiate:

$mx_{ttt} + k_d x_{tt} + k_p x_t + k_i x = 0$ ➔ settles to $x = 0$!

# The PID

$$C = k_p + k_d s + k_i/s$$
$$= (k_p s + k_d s^2 + k_i) / s$$

High-frequency response is ~$k_d s$; increases with frequency and disobeys our prior rule about infinite power. High frequency errors will lead to very large control action!




Sensor noise solutions:

- use a very clean and high-res. sensor for x, which can be easily differentiated numerically, *e.g., motor encoder*
- use a sensor that measures xt directly, *e.g., tachometer*
- filter the measurement. For a low-pass, we would get

$$C_f = [ (k_p s + k_d s^2 + k_i) / s ] \times [ \lambda / (s+\lambda) ]$$
$$= \lambda (k_p s + k_d s^2 + k_i) / s (s+\lambda)$$

But combine with a double integrator plant $P = m/s^2$

PC = $m(k_p s + k_d s^2 + k_i) / s^3$, which *does* go to zero at high frequencies, as desired → the system does have a real bandwidth, which can be tuned.