

A. Selected formulae

Time-dependent Schroedinger equation:

$$i\hbar \frac{\partial}{\partial t} \Psi(x, t) = \hat{H} \Psi(x, t)$$

Time-independent Schroedinger equation:

$$\hat{H} \psi(x) = E \psi(x)$$

Position representation of Hamiltonian operator

$$\hat{H} = -\frac{\hbar^2}{2m} \left(\frac{\partial}{\partial x} \right)^2 + V(x)$$

Position representation of momentum operator

$$\hat{p} = \frac{\hbar}{i} \frac{\partial}{\partial x}$$

Momentum representation of position operator

$$\hat{x} = i\hbar \frac{\partial}{\partial p}$$

Probability current

$$J(x, t) = \frac{\hbar}{2im} \left(\Psi^*(x, t) \frac{\partial \Psi(x, t)}{\partial x} - \frac{\partial \Psi^*(x, t)}{\partial x} \Psi(x, t) \right)$$

Fourier transform

$$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x) e^{-ikx} dx$$

Inverse Fourier transform

$$\psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k) e^{ikx} dk$$

Heisenberg uncertainty

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

Uncertainty of a quantity

$$(\Delta x)^2 = \langle x^2 \rangle - \langle x \rangle^2$$

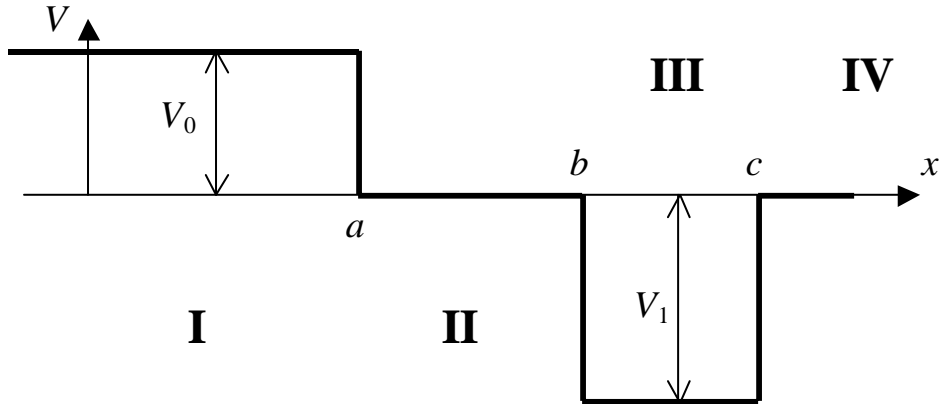
Infinite potential well

$$V(x) = \begin{cases} 0 & \text{for } 0 \leq x \leq a \\ \infty & \text{elsewhere} \end{cases}$$

$$E_n = \frac{\hbar^2 \pi^2}{2ma^2} n^2$$

$$u_n(x) = \sqrt{\frac{2}{a}} \sin(n\pi \frac{x}{a})$$

1. **Wavefunctions for piecewise constant potential** (20 points).



Consider the scattering problem with the potential sketched above, and take the energy E of a particle of mass m to be less than the potential V_0 . ($0 < E < V_0$.)

- (5 points) Without actually solving the Schrodinger equation, set up the solutions in the different regions I-IV if particles are only incident from the right. Relate the coefficients appearing in your expressions for the wavefunctions in the different regions to \hbar , m , E , V_0 , and V_1 . (Do not take into account boundary conditions yet.)
- (5 points) Write down the equations that specify the boundary conditions at $x=c$.
- (5 points) Instead, assume that bound states of energy $E_b < 0$ exist in the potential well $b \leq x \leq c$. Set up the solutions in the different regions in this case, and relate the coefficients appearing in your expressions for the wavefunctions to \hbar , m , E_b , V_0 , and V_1 . (Do not take into account boundary conditions.)
- (5 points) Sketch the wavefunction for the first excited state in the well if you assume that
 - the first excited state is just barely bound;
 - the first excited state is deeply bound.

What quantitative condition on the binding energy E_b or on V_1 distinguishes the two cases?

2. Time evolution of wavefunction in infinite box potential (30 points).

Consider a particle of mass m inside a box of size a with infinite walls,

$$V(x) = \begin{cases} 0 & \text{for } 0 \leq x \leq a \\ \infty & \text{elsewhere} \end{cases}.$$

The wavefunction is specified at $t=0$ to be

$$\psi(x, t=0) = C(3 \sin 2kx - 2 \sin 3kx), \text{ where } k = \pi / a.$$

- a) (5 points) Determine the normalization coefficient C .
- b) (5 points) Expand the wavefunction at the initial time $\psi(x, t=0)$ in terms of eigenfunctions of the infinite box, i.e. determine the expansion coefficients c_n .
The eigenfunctions are given in the formula section of this exam.
- c) (5 points) Write down $\psi(x, t)$ at an arbitrary later time t .
- d) (7 points) If a measurement of the particle's energy at time t is performed, what will be the possible outcomes, and with what probability will those values be measured? What is the average energy $\langle E \rangle$ of the particle in the box? Is $\langle E \rangle$ changed by the measurement?
- e) (8 points) Calculate the probability current $J(x, t=0)$ at the initial time. Does it depend on the complex phase that you have chosen for C , and why (not)? Will the value of J remain unchanged at later times? Express the fact that the particle does not leave the box as a mathematical condition on J .

3. Square wavefunction in momentum representation (25 points)

The wavefunction of a particle of mass m in free space ($V=0$) is given at time $t=0$ in the wavevector representation by the expression

$$\phi(k) = \begin{cases} C & \text{for } -k_0 \leq k \leq k_0 \\ 0 & \text{elsewhere} \end{cases} .$$

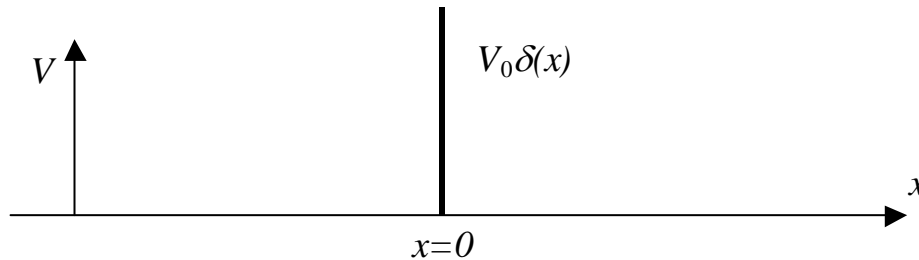
- a) (2 points) Find a normalization constant C .
- b) (6 points) Calculate Δp for the above wavefunction.
- c) (3 points) Write down an expression for the wavefunction in position space $\psi(x, t=0)$ at time $t=0$ without actually evaluating the integral.
- d) (5 points) Write down an expression for the wavefunction in position space $\psi(x, t)$ at arbitrary time t without actually evaluating the integral, and explain in one sentence why you can write $\psi(x, t)$ in that form.
- e) (9 points) Using the expression from part b), calculate $\psi(x, 0)$ and sketch the probability density in position space.

4. **Scattering by a repulsive δ -potential** (25 points)

A beam of particles, each of mass m and energy $E > 0$, is incident on a repulsive delta potential located at the origin,

$$V(x) = V_0 \delta(x),$$

with $V_0 > 0$.



- a) (3 points) Write down the wavefunctions in the regions $x < 0$ and $x > 0$ for the stationary scattering problem with a particle incident from the left.
- b) (3 points) What are the units of the constant V_0 ?
- c) (4 points) Is it true that $R+T=1$ in this problem, where R is the reflection and T is the transmission coefficient, defined as the fraction of reflected and transmitted particles, respectively? How would you have to modify the potential to obtain a different answer?
- d) (5 points) Write down the net current in the region $x < 0$ in terms of the incident amplitude A , the particle energy E , mass m , the reflection coefficient R , and the transmission coefficient T , and fundamental constants. Normalize the wavefunction such that the incident particle density in the region $x < 0$ corresponds to one particle per deBroglie wavelength. What is the corresponding incident current?
- f) (10 points) Calculate the fraction of particles in the incident beam that are reflected by this repulsive potential, i.e. find the reflection coefficient R . Write your answer in terms of E , m , V_0 , and fundamental constants. Check if your result is reasonable in the limit $V_0 \rightarrow 0$ for constant E .