

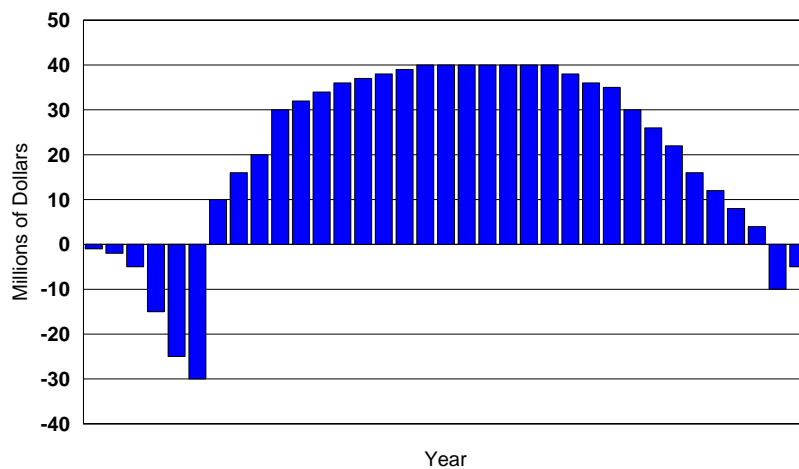
Notes on Project Evaluation

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Note on Equivalence of Cash Flows

In evaluating projects, we often have many alternatives to consider. For each alternative, we generally try to predict cash flows over a long time horizon, taking into account the costs of construction, the continuing costs of maintenance, and the costs and revenues related to operations. The alternatives that we are investigating may have sizable differences in terms of investments, construction costs, performance capabilities, and projected operating costs and revenue potential. Comparisons will not be trivial. We therefore would like to be able to convert these complex cash flows into something that is equivalent and easily understood that will allow us to rank the alternatives in terms of financial measures.

Cash Flow of a Typical CEE Project



Equivalence is a key concept. Two projected streams of cash flows are equivalent for me if they are equally acceptable to me, i.e. if I am indifferent whether I receive one or the other. The equivalence of cash flows is related to the time value of money, the investment opportunities that are available, and the perceived risks associated with the project during the time period in question, as will be discussed below.

Our goal therefore is to transform an arbitrary stream of cash flows into an equivalent cash flow that is “easily understood”. Three financial measures are commonly used:

1. P: the equivalent present value of the cash flows (what is the projected stream of cash flows worth to me today?)
2. F: the equivalent future value of the cash flows (what is it worth to me some time in the future?)
3. A: the equivalent annuity amount (what is it worth to me in terms of receiving a uniform cash flow of A per period for N periods).

P, the present value of the cash flows, is clearly the easiest to understand. If we have a choice among various alternatives, each of which is equivalent to getting a lump sum of money today, we will simply pick up the biggest pile of money (assuming for now that money is our chief object in life and that the money related to the various projects is indeed legally acquired!).

The other two measures are also easy to understand. If our time of reference is some time in the future, we will presumably want the alternative with the maximum future value. If we are more comfortable dealing with monthly or annual cash flows, then we can express options in terms of an annuity and choose the one that pays the most.

The basic tasks in financial analysis of a project can be summarized as follows:

- Predict the cash flows over the life of the project
- Estimate the present value of the cash flows
- Calculate the equivalent future value or annuity, if desired
- Rank projects by P, F, or A (since they are equivalent measures, the ranking will be the same)

To convert future cash flows into a present value requires consideration of the time value of money. To begin, let’s assume that we are a private company considering various projects that we can finance by borrowing from a bank at an interest rate of $i\%$. For simplicity, assume that we can also put money into a savings account at the bank and also receive $i\%$ interest. Our choices therefore are either to invest in more projects or to put money into our bank account. If a project earns more than $i\%$, then we will invest; if it earns less than $i\%$, we’d be better off putting money in the bank.

Let’s begin with the simplest possible cash flows: the project requires an investment of P at time 0 and generates F at the end of period N. Our alternative investment is to put the money in the bank, in which case our investment will increase by a factor of $(1 + i)$ per period. By the end of period N, our bank account will have grown to $P * (1+i)^N$. Thus, we will prefer to invest if F is greater than this, and we will prefer to put the money in the bank if F is projected to be less than this. To compare values for time 0 and for time N, we need to use the factor $(1+i)^N$ to “discount” the future value. “Discount” comes from a Latin term that meant “count for less”. The process of discounting, therefore, is the process of reducing the current value of future cash flows to take into account the time value of money and the uncertainties that may exist between today and tomorrow.

It is useful to introduce some notation for the discount factors. $[F/P, i, N]$ can be used to denote the factor that calculates the future value F as a fraction of the present value P , assuming a discount rate of $i\%$ for N periods. $[P/F, i, N]$ denotes the factors that is used to determine the present value P given the future value F , again assuming a discount rate of $i\%$ for N periods. The two factors are used as follows to compare future and present values:

$$F(N) = P * [F/P, i, N] = P * (1+i)^N$$

$$P = F * [P/F, i, N] = F / (1+i)^N$$

If you have a spreadsheet, you can use these equations repeatedly to convert an arbitrary cash flow into either a present or a future value. Since an annuity of A per period is certainly one possible cash flow, you can convert an annuity to either a present or future value using these equations. If you make interest rate, annuity amount, and the number of periods a variable, you can easily find the annuity amount that is equivalent to any present or future value. However, it can be more elegant (and less time-consuming) to use algebraic expressions to convert P or F into annuities (or to convert annuities into P or F).

PV of \$1.00 Received at Time t

	5 Yrs	10 Yrs	20 Yrs	50 Yrs	100 Yrs
1%	0.95	.91	0.82	0.61	0.37
5%	0.78	0.61	0.38	0.088	0.0076
10%	0.62	0.038	0.15	0.0085	0.000072
20%	0.40	0.16	0.026	0.00011	0.00000001

Calculating the Future Value of Annuity $[F/A, i, N]$

$[F/A, i, N]$, which is called the “uniform series compound amount factor”, can be used to find the future value F that is equivalent to an annuity of A /period for N periods assuming an interest rate of I . We assume that a payment of A is made at the end of each period and that each payment is

invested at $i\%$ per period for the remaining period. The future value will be proportional to A , and $[F/A, i, N]$ will be the proportionality factor:

$$F(N) = A * [F/A, i, N-1]$$

It is possible to find a compact expression for $[F/A, i, N]$ by using well-known relationships for geometric sequences. First, we can calculate F just by converting each payment $A(t)$ to a value at the end of the N th period. The payment at the end of period t will earn interest for another $N-t$ periods, so its future value will be:

$$A(t) = A * (1+i)^{N-t}$$

If we add them all up, we will get

$$F(N) = A * ((1+i)^{N-1} + (1+i)^{N-2} + \dots + (1+i)^{N-t} + \dots + (1+i)^0)$$

If we let $b = 1+i$, this is equivalent to a simple geometric sequence:

$$F(N) = A * (b^{N-1} + b^{N-2} + \dots + b^{N-t} + \dots + b^0)$$

If we rearrange the terms, then

$$F(N) = A * (1 + b + \dots + b^{N-t} + \dots + b^{N-1}) = A * [F/A, i, N]$$

and

$$[F/A, i, N] = (1 + b + \dots + b^{N-t} + \dots + b^{N-1})$$

Now use a mathematical trick: multiply this by $(1-b)/(1-b) = 1$ to get a more elegant result:

$$\begin{aligned} [F/A, i, N] &= 1/(1-b) * [(1 + b + \dots + b^{N-t} + \dots + b^{N-1}) - (b + b^2 + \dots + b^{N-t+1} + \dots + b^N)] \\ &= 1/(1-b) * [1 - b^N] \end{aligned}$$

Substitute $(1+i) = b$ and rearrange terms to get the uniform series compound amount factor:

$$[F/A, i, N] = [(1+i)^N - 1]/i$$

Expressions for the Other Factors

Expressions for the other factors can readily be found. Since $F(N) = A * [F/A, i, N]$ we can invert the above to get $[A/F, i, N]$, which is known as the sinking fund factor:

$$[A/F, i, N] = i/[(1+i)^N - 1]$$

Since $F(N) = P * (1+i)^N$, we can immediately write

$$F(N) = [F/A, i, N] = P * (1+i)^N$$

so that the “uniform series present worth factor” is

$$[P/A, i, N] = [F/A, i, N]/(1+i)^N = [(1+i)^N - 1]/[i (1+i)^N]$$

The inverse of this expression will give the capital recovery factor:

$$[A/P, i, N] = [i (1+i)^N]/[(1+i)^N - 1]$$

These expressions may not seem too elegant! But note that when N gets large, they become very clear and simple:

$$[P/A, i, N] = 1/i$$

$$[A/P, i, N] = i$$

These are very useful relationships to keep in mind, as they allow easy, approximate conversions between annuities and present value.

Nominal vs. Effective Interest Rates

The nominal rate is the interest rate that you would receive if interest were compounded annually. With a 12% nominal rate, an investment of \$1000 on January 1 would earn \$120 interest on December 31. However, if you compound interest semiannually, then you would get more interest. First after 6 months, you would earn 6% or \$60 on your initial investment. Then, at the end of the year, you would earn another \$60 on the initial investment plus \$3.60 on the interest that you received at the end of June. For the year, you therefore earned 12.36% interest on your investment, even though the nominal rate was 12%.

More rapid compounding gives a bit higher effective rate, but there are diminishing returns:

- Quarterly rate is 12.55%
- Bimonthly rate is 12.62%
- Monthly rate is 12.68%
- Daily rate is 12.75%

Clearly, this is approaching a limit. To find this limit, let's first write an algebraic expression for what we are doing as we compound more frequently. If r is the nominal interest rate, but we compound M times per year, then the effective rate will be

$$i = [1 + (r/M)]^M - 1$$

and the factor $[F/P, r\%, 1]$ will be

$$[F/P, \underline{r}\%, 1] = [1 + (r/M)]^M$$

Note that the term “ $\underline{r}\%$ ” represents the nominal interest rate r and the use of continuous compounding.

If we let $p = M/r$ and rewrite this equation, we get

$$[F/P, \underline{r}\%, 1] = (1 + 1/p)^{rp} = ((1 + 1/p)^p)^r$$

This is a classic relationship, as the limit of $(1+1/p)^p$ as p approaches infinity is $e = 2.7128\dots!$ (For a derivation, see a calculus text; in my ancient bookshelf, I found this in Apostle, **Calculus**, Vol. 1, 1st edition, 1961, p. 405. He notes that “sometimes this limit relation is taken as the starting point for the theory of the exponential function”.)

Thus, we have

$$[F/p, \underline{r}\%, 1] = e^r$$

and therefore

$$[F/P, \underline{r}\%, N] = e^{rN}$$

which somewhat unexpectedly gives us a very nice relationship:

$$[F/P, \underline{r}\%, N] = e^{rN} = (1 + i)^N$$

where the exponential expression assumes continuously compounding using the nominal rate and the other expression uses the effective rate. Hence, we can revise all of the formulas for discrete cash flows into expressions for continuous compounding by substituting $e^{rN} = (1 + i)^N$.

This is more than a way for considering nuances and technicalities in calculating interest rates. The continuous compounding formulation is also very useful, because it is easy to remember and forms the basis for some quick approximations. All you need is to remember some useful values of rN :

- If $rN = 1$, $e^{rN} = 2.718\dots$
- If $rN = 0.7$, $e^{rN} = 2.013\dots$, approximately 2
- If $rN = 1.1$, $e^{rN} = 3.004\dots$, approximately 3
- If $rN = 1.4$, $e^{rN} = 4.055\dots$ approximately 4

You can use these relationships to figure out how long it will take double or triple your money. If the nominal interest rate is 10%/year, then it will take 7 years to double your money and 11 years to triple your money. You can also use the inverse relationship to calculate present values. For example, if the nominal interest rate is 7%, then the present value of something received in 7 years will be half its future value. In other words, it is possible to make mental estimates of what you probably thought were quite complex functions. I realize that mental math has been an increasingly undervalued skill since the invention of the electronic calculator, but I guarantee

that you will find it useful to use the above relationships if you are ever involved in face-to-face discussions, negotiations, or debates related to project evaluation when it would be inconvenient or inappropriate to use your calculator or computer. You can do present value analysis in your head - and that will give you an advantage in negotiation!