MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Civil and Environmental Engineering

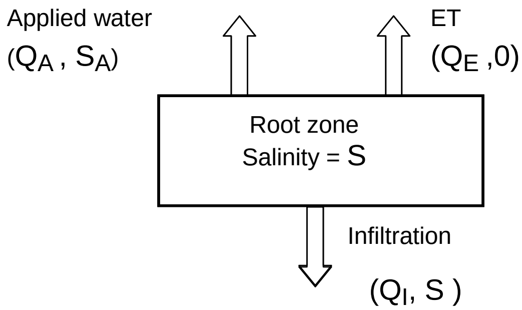
1.020 Ecology II: Engineering for Sustainability

Problem Set 2 – Mass Conservation, Box Models, Stability
Due: 12 noon Thursday Feb. 22, 2007

---

1. Soil Salination (20 points)

Consider the column of soil below a unit area of cropland (see sketch below). The crop is irrigated with water that has salinity S_A (mg/L). The water application (inflow) rate is Q_A L/day.



Q_I L/day of the irrigation water moves through the root zone and infiltrates into the deep soil below. The infiltration water has the same salinity S as the water in the root zone. Q_E L/day is transpired by the crop or evaporated from bare soil. The evaporated water contains no salt (S_E = 0 mg/L). The applied and evapotranspired water are related by the irrigation efficiency e, which is defined as:

$$e = \frac{Q_E}{Q_A}$$

a) Write steady-state water and salt mass balance equations for the root zone (the control volume), assuming that the irrigation efficiency is known.

b) Derive an expression that relates the steady-state salinity S mg/L to the irrigation efficiency, for given values of Q_A and S_A. What happens to the root zone salinity as the irrigation efficiency approaches 1? What does this imply about the design of irrigation systems?

2. Ocean Acidification -- Equilibrium Analysis (20 points)

Background

Equilibrium chemistry is an important application of mass and charge balance principles. In this problem you use equilibrium chemistry calculations to evaluate the effects of increased atmospheric CO_2 on pH in the oceans. The analysis is very simplified in order to keep the problem from becoming too difficult. However, it captures some of the important features of equilibrium chemistry.

Gaseous carbon dioxide CO_2 upon dissolving into water becomes carbonic acid H_2CO_3. H_2CO_3 is a weak acid that exists in equilibrium with both carbonate CO_3^{2-} and bicarbonate HCO_3^- ions.

Problem statement

Assume that the dissolved CO_2 (in the form of H_2CO_3) has equilibrated with atmospheric CO_2. Under standard temperature and pressure and today's atmospheric CO_2 levels the H_2CO_3 concentration at equilibrium is 1.23 x 10^{-5} moles per liter (M), or $10^{-4.91}$M (You will see below why it is easier to express it in log units). Using square brackets to represent concentrations in M this atmosphere-water equilibrium condition becomes:

$$[H_2CO_3] = 10^{-4.91} \quad (1)$$

The H_2CO_3 and H_2O undergo the following reactions:

$$H_2CO_3 \leftrightarrow H^+ + HCO_3^- \qquad pKa_1 = -6.35$$

$$HCO_3^- \leftrightarrow H^+ + CO_3^{2-} \qquad pKa_2 = -10.33$$

$$H_2O \leftrightarrow H^+ + OH^- \qquad pKa_3 = -14$$

These equations define mass balances for the species H_2CO_3, HCO_3^-, CO_3^{2-}, H^+ and OH^-. The mass balances are supplemented by the equilibrium relationships, which may be written as:

$$\frac{[H^+][HCO_3^-]}{[H_2CO_3]} = 10^{-6.35} \quad (2)$$

$$\frac{[H^+][CO_3^{2-}]}{[HCO_3^-]} = 10^{-10.33} \quad (3)$$

$$[H^+][OH^-] = 10^{-14} \qquad (4)$$

The equilibrium coefficients given here apply at standard temperature and pressure:

In a neutrally charged environment, the total charge must also be conserved. This can be expressed in this system as

$$[H^+] = [HCO_3^-] + 2[CO_3^{2-}] + [OH]^- \qquad (5)$$

a) Use the equilibrium constraints and charge balance given in (1) – (5) to derive a cubic equation for the single unknown $[H^+]$, and solve for $[H^+]$ (either by hand, with a calculator, or with MATLAB's `fzero` function). What is the pH of the system, where pH = - $\log_{10}[H^+]$?

b) If the level of CO_2 in the atmosphere is allowed to double, and the temperature and remains the same, the equilibrium concentration of carbonic acid $[H_2CO_3]$ will also double. What is the resulting pH in the ocean?

3. Biogas Generation (25 points)

Biogas production generates methane gas CH_4 from anaerobic bacterial digestion of human and animal waste. A stoichiometric equation describing this conversion can be written as follows:

$$C_{10}H_{20}O_6N_2 + 3H_2O \rightarrow 5.5CH_4 + 4.5CO_2 + 2NH_3$$

where $C_{10}H_{20}O_6N_2$ is an approximate formula for chemical composition of the waste biomass material.

The methane gas produced can be collected and burnt as fuel:

$$CH_4 + 2O_2 \rightarrow CO_2 + 2H_2O$$

At standard temperature and pressure (25C, 1 atm) this reaction produces 891 kJ of energy per mole of CH_4 burned.

Using the above equations, consider the following:

a) In a rural village where 100 kg of biomass is collected daily and placed in a batch digestor, calculate the amount of energy production possible from the methane produced daily. If this methane is to be used in a generator for 12 hours a day, and the generator can convert 70% of the combustion energy into electrical energy, calculate the average power (in watts) obtainable from biogas for the village.

b) The MBTA buses run on biogas as fuel, in the form of compressed natural gas (CNG, primarily liquid methane). If biomass were the only source of this methane, what amount of raw biomass is required to fill a 95 gallon CNG tank? (The density of CNG is about 500 kg/m3).

4. Fisheries Modeling (35 points)

Background

Fisheries modeling is an active and controversial topic since it is often used to justify policy positions and to determine whether or not a fishery is endangered. In this problem we consider a very simple fisheries model that describes the interaction between the fish stock and the fishing industry. This model considers mass conservation (in a fish biomass equation) but also considers financial accounting (in a capital equation).

Suppose this fish stock (biomass) N has a net growth rate r. The fisheries production (catch) P is written using standard economic assumptions as:

$$P = \sqrt{KL}N \qquad (1)$$

where K is the capital invested and L is the labor force. The model is based on the following equations for the fish stock and capital:

$$\frac{dN}{dt} = rN - a_n P \qquad (2)$$

where r is the net unstressed growth (replenishment) rate of the fish population and a_n relates production to the number of fish harvested.

$$\frac{dK}{dt} = (1 - a_c)(a_k P - \mu K) \,. \qquad (3)$$

where $a_k P$ is the revenue accumulation rate, μK is loss of capital through depreciation, and $a_k P - \mu K$ is the net revenue accumulation rate. The coefficient a_c is the fraction of net revenue accumulation that is diverted to consumption so $1 - a_c$ is the fraction remaining for capital investment. When dK/dt is positive more money is invested in fishing and the fleet increases in size or efficiency. By combining mass balance and economic considerations this model allows us to relate the dynamics of a fisheries resource to the amount of capital invested. The model is nonlinear since harvesting and revenue accumulation are both proportional to $\sqrt{K}N$.

A nontrivial steady-state solution (i.e. a solution with $\tilde{N} \neq 0$ and $\tilde{K} \neq 0$) is given by $N = N^*$ and $K = K^*$, where:

$$N^* = \frac{\mu r}{L a_n a_k}; \quad K^* = \frac{1}{L}\left(\frac{r}{a_n}\right)^2 \tag{4}$$

If we define a new set of nondimensional variables we can reduce the number of parameters in our model so we can more easily examine its behavior. To do so we let

$$\tilde{N} = \frac{N}{N^*}; \; \tilde{K} = \frac{K}{K^*}; \quad \tau = rt \, . \tag{5}$$

Substituting (4) and (5) into (2) and (3) and using (1) we arrive at the following non-dimensional set of 2 coupled nonlinear ODEs:

$$\frac{d\tilde{N}}{dt} = \tilde{N}\left(1 - \sqrt{\tilde{K}}\right) \tag{6}$$

$$\frac{d\tilde{K}}{dt} = \beta\left(\tilde{N}\sqrt{\tilde{K}} - \tilde{K}\right) \tag{7}$$

where

$$\beta = \frac{(1 - a_c)\mu}{r} \tag{8}$$

The parameter β is the ratio of capital depreciation to resource replenishment.

Problem statement

a) Derive a non-trivial steady-state solution to (6) and (7).

b) Evaluate the Jacobian matrix (community matrix) at the non-trivial steady-state solution you obtained in a). Derive the eigenvalues of this matrix and use them to determine the stability of the fisheries model. You can carry out this derivation by hand or you can use the MATLAB `eig` function. What do the eigenvalues tell you about the temporal response of the model?

c) Using β = 0.1 and the initial conditions $\tilde{N}_0 = 1$ and $\tilde{K}_0 = 0.5$, write a MATLAB code that uses a differential equation solver (e.g., `ode23`) to find $\tilde{N}$ and $\tilde{K}$. Plot $\tilde{N}$ and $\tilde{K}$ as functions of time. Also plot $\tilde{N}$ versus $\tilde{K}$ (this is called a "phase portrait").

d) Increase β to a value of 0.2 to examine the behavior of a fishery with a lower replenishment rate (or, equivalently, a higher depreciation rate). Find the new solution and generate a new set of plots.

e) Summarize the effect of β.

f) [Optional] Multiply β by random numbers in order to account for natural temporal fluctuations in economic and environmental conditions. To do this, use the command `rand` to generate an array of random numbers (distributed uniformly between 0.0 and 1.0) for all the time steps defined in the argument of the `ode23` function. Add 0.5 to each random number in the array so the resulting adjusted array values vary between 0.5 and 1.5 (this gives an average β multiplier of 1.0). Insert the entire random array as a time-dependent multiplier of β in the right-hand side of the differential equation. This can be done as follows, where `dX(2)` represents the right-hand side of (7), `X(1)`represents $\tilde{N}$ and `dX(2)`represents $\tilde{K}$:

```
dX(2)=beta*(rndR(round(t)+1)*sqrt(X(2))*X(1)-X(2));
```

The `round` function is needed to insure that the `t` value generated by `ode23` is converted to an integer array index.

Now re-solve the equations and re-plot your results for $\beta = 0.1$.

Summarize the effect of the random fluctuations.