**MASSACHUSETTS INSTITUTE OF TECHNOLOGY**
**Department of Civil and Environmental Engineering**

**1.020 Ecology II: Engineering for Sustainability**

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**Problem Set 5 –Thermodynamics and Climate Model**
**Due: 5PM Monday April 9, 2007**

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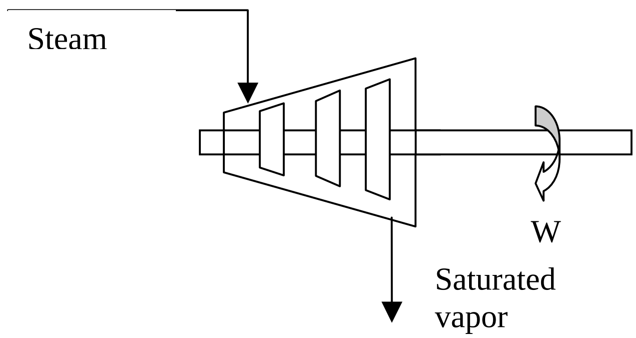
## Problem 1: Steam Turbine (15 points)

A turbine is a device that generates work by forcing a stream of high pressure gas to turn its blades. The high pressure gas stream loses its pressure on its way out, transferring its kinetic energy to the rotational energy of the turbine shaft.




In the above example, 1 kg/s of hot superheated steam at 300 C is fed into the steam turbine at 2.5MPa. The turbine is adiabatic. The exiting stream is at 0.1MPa, and is a saturated vapor at 100 C (just about to contain some condensed water). If 70% of the turbine rotational energy is used to drive a generator to convert to electrical energy, calculate the electrical output of this system.

You can find tables for system properties of water (such as enthalpy) at:

http://www.spiraxsarco.com/esc/SH_Properties.aspx (superheated conditions)

http://www.efunda.com/materials/water/steamtable_sat.cfm (saturated conditions)

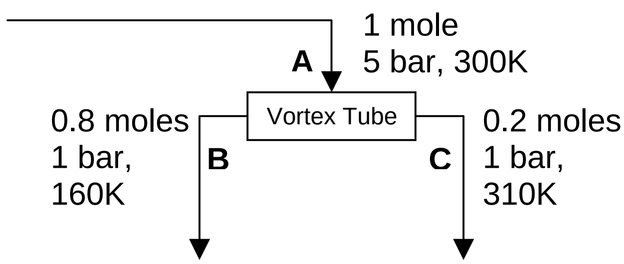
Similar tables are also available in many thermodynamic s texts, including the text reserved for this class:

## Problem 2: Vortex Tube (20 points)

A Hilsch vortex tube is a device that can separate a pressured stream of gas into a hot and cold stream. It is a mechanical device with no moving parts – the gas enters at a high pressure, and is forced through spiral tube at high speed. At the end of the spiral, some of the gas is allowed out, but the rest is made to bounce back. The sudden change of kinetic energy in one stream generates a heat transfer to the other gas stream, resulting in a hot and cold stream of gas.

Before the advent of modern refrigeration it was thought that this sort of device would be a good way to obtain very cool streams of gas for cooling purposes. In particular, if the initial stream originates from a waste gas stream from another industrial process, the vortex tube could make use of this 'free waste' pressure that would otherwise be vented to the surroundings.

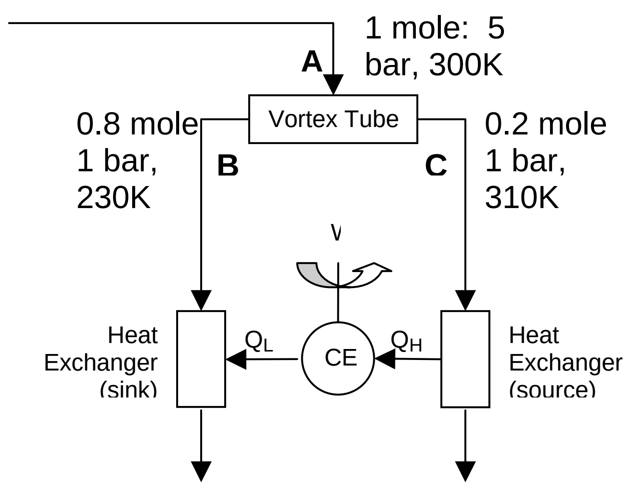
A vortex tube is fed with air at 300 K and 5 bar into a tangential slot near the center (Denoted A). Consider a packet of 1 mole of ideal gas at A entering the system. The packet splits into two, creating two air streams. The engineer responsible for designing the tube maintains that the two output air streams will have the properties indicated in the diagram below.




**Part a) (5 points):** Calculate the total change in entropy imparted on the gas by the vortex tube. Assume ideal gases that have a constant heat capacity of $c_p$ = 29.3 J/mol K. Are the output pressures and temperatures promised by the engineer consistent with the second law?

**Part b) (5 points):** Repeat a) with the temperature of B changed to 230 K.

Now we propose to operate a Carnot engine that will utilize the temperatures of the two split streams from b) to generate shaft work (see Figure below).



The two packets of air leaving the vortex tube pass through heat exchangers that act as a high and low temperature reservoirs for a Carnot engine (CE).

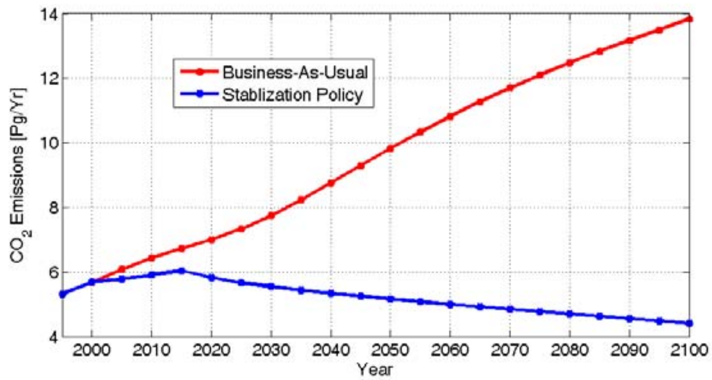
**Part c) (5 points):** Calculate the Carnot efficiency of the engine.

**Part d) (5 points):** Calculate the maximum work that the Carnot engine extracts from the one mole of gas injected at A.

**Problem 3: Dynamic Three-Component Model of the Earth Energy Balance (65 points)**

Soon you will engage in the 'Wedge-Game', where you will investigate alternative ways to control $CO_2$ emissions. The target of the Wedge-Game is to stabilize the atmospheric concentration of $CO_2$ in the 550-650 ppm (parts per million) range. Pre-industrial levels were around 270 ppm and we stand at 380 ppm today.

In order to reach this stabilization goal in the next 100 years, the combustion of fossil fuels needs to be radically cut back. MIT economists and climate scientists at MIT (see http://web.mit.edu/globalchange/www/) estimate that the future fossil-fuel $CO_2$ emissions in petagrams (1 Pg = $10^{15}$ g) must adjust as shown by the blue line in the figure below:




This is a dramatic change and will have significant socioeconomic consequences. If we proceed with business-as-usual, the emissions will grow dramatically, as shown by the red line. In either case the $CO_2$ concentrations in the atmosphere will build up and affect global temperatures.

In PS3 you worked with the following earth energy balance model:

$$\frac{dT}{dt} = F_{nf}(T) = \frac{1}{K}\left[\frac{\Omega}{4}(1-a) - \sigma T^4\right] \qquad (1)$$

In particular, you solved for the equilibrium temperature $T_e \approx 255$ °K that satisfies the steady-state condition $F_{nf}(T_e) = 0$. That temperature is too cold. The reason is that greenhouse gases (principally $H_2O$ and $CO_2$) absorb the outgoing thermal radiation and radiate it back down. As a result, the overall Earth equilibrium temperature must rise in order to maintain the overall thermodynamic equilibrium of the Earth system. The absorption is about 40% of the thermal radiation. We represent that with $\varepsilon = 0.4$ for current climate so that we write the Earth energy balance with feedback as:

$$\frac{dT}{dt} = F_f(T) = \frac{1}{K}\left[\frac{\Omega}{4}(1-a) - (1-\varepsilon)\sigma T^4\right] = \frac{1}{K}G(T) \qquad (2)$$

where $G(T)$ [$Wm^{-2}$] contains the radiative exchanges.

**Part a) (10 points):** What is the new Earth equilibrium temperature when the greenhouse effect is included?

A sensitivity coefficient $\lambda$ [in $Wm^{-2}K^{-1}$] can be defined that gives the linear response of the system to the temperature state as follows:

$$G(T) \approx G(T_e) + \lambda(T - T_e) \qquad (3)$$

**Part b) (5 points):** Estimate $\lambda$ using a Taylor-series approximation.

A key feature of the Earth system that may produce problems if we delay stabilization policies is that the system has large thermal inertia due to the presence of large oceans. Water has a very high heat capacity and the oceans are big.

In order to examine this effect we separate the Earth into one atmospheric and two ocean compartments, each with its own energy balance and internal energy. Atmospheric variables (e.g. temperature) will be indicated with "*a*" subscripts. The ocean is divided into upper (mixed) and lower (deep) compartments, indicated by "*m*" and "*o*" subscripts, respectively. The mixed ocean compartment corresponds to the upper 100 m mixed layer that is linked to the atmosphere through energy fluxes. The deep ocean compartment corresponds to the ocean below 100 meters depth. The mixed ocean compartment also exchanges heat with the deeper ocean. The energy flux $F_{ma}$ between the mixed ocean and atmosphere and the energy flux $F_{om}$ between the deep ocean and mixed ocean are measured in [$Wm^{-2}$] and are each proportional to temperature differences, as follows:

$$F_{ma} = \lambda_{ma}(T_m - T_a) \qquad (4)$$
$$F_{om} = \lambda_{om}(T_o - T_m) \qquad (5)$$

**Part c) (10 points):** Write a system of coupled linear energy balance equations for the atmosphere, mixed ocean, and deep ocean compartments, using the linear approximation of (3) in the atmosphere equation and the flux relationships (4) and (5) where applicable. Denote the per unit area heat capacities of the three heat reservoirs as $K_a$, $K_m$, and $K_o$ [Wyr $m^{-2}K^{-1}$]. Express this system in the form $\frac{d\boldsymbol{T}}{dt} = \boldsymbol{AT} + \boldsymbol{u}$, where the 3 dimensional state vector is $\boldsymbol{T} = [T_a \quad T_m \quad T_o]^T$ and the "community matrix" $\boldsymbol{A}$ is a 3 by 3 matrix that depends on the parameters $K_a$, $K_m$, $K_o$, $\lambda_{om}$, $\lambda_{ma}$, and $\lambda$.

Since the system of equations is linear the deviation (or perturbation) $\Delta \boldsymbol{T}$ of the temperature state from a nominal value (e.g. an equilibrium value) satisfies the following perturbation equation:

$$\frac{d\Delta \boldsymbol{T}}{dt} = \boldsymbol{A}\Delta \boldsymbol{T} + \Delta \boldsymbol{u} \qquad (6)$$

The $\Delta \boldsymbol{u}$ vector contains perturbation forcing terms. If the original system is initialized at the nominal value used to define the perturbation the initial condition for the perturbation equation is $\Delta \boldsymbol{T}(0) = \mathbf{0}$.

**Part d) (20 points):** Construct a numerical model of the global ocean-atmosphere temperature perturbations defined by (6). You may use either the MATLAB differential equation solver `ode23b` or a closed-form analytical solution (**extra credit 15 points**) in your program.. The projected $CO_2$-induced perturbation forcing terms $\Delta \boldsymbol{u}$ [$Wm^{-2}$] appearing in (6) are provided (for both emissions scenarios) in the MATLAB file `Forcing.mat` (this file is available on the 1.020 Stellar site).

**COMPUTING HINT:** `Forcing.mat` includes 1 array (`Forcing_year`) listing the forcing years and 2 arrays of annual time series (`NoPolicy_Forcings_Wm2` and `Policy_Forcings_Wm2`) for the period 1995-2100. The data in the file may be loaded directly into MATLAB arrays in your program with the MATLAB `load` function (see MATLAB help for details). Be sure to put the data file in the same directory as your program. You can look in your workspace to check the names and sizes of the two arrays contained in `Forcing.mat` if you type `load Forcing.mat` in the command window. Then insert the `load` command directly into your program and access the arrays in `Forcing.mat` within the program using the appropriate array names.

**CLOSED FORM SOLUTION HINT**: If you select the closed form approach, you may find it beneficial to use the change of variables $\boldsymbol{x} = \Delta \boldsymbol{T} + \boldsymbol{A}^{-1}\Delta \boldsymbol{u}$ and the integration factor $e^{t\boldsymbol{A}}$ for the integration over a year increment $t \in [0, \Delta t]$. The matrix exponential $e^{t\boldsymbol{A}}$ needs to have a diagonal matrix as its argument. So when it comes to actually evaluating it, note that:

- $\boldsymbol{A} = \boldsymbol{VDV}^{-1}$ is the eigen-decomposition of the matrix $\boldsymbol{A}$. $\boldsymbol{V}$ contains the eigenvectors and $\boldsymbol{D}$ is the diagonal matrix of eigenvalues (for non-repeated eigenvalues). You can perform the eigen decomposition with the `eig` function in MATLAB.
- If $\boldsymbol{V}$ is orthonormal then $e^{\boldsymbol{VDV}^{-1}} = \boldsymbol{V}e^{\boldsymbol{D}}\boldsymbol{V}^{-1}$. You can prove this by expanding $e^{\boldsymbol{VDV}^{-1}}$ in a Taylor series, factoring $\boldsymbol{V}$ out on the left, factoring $\boldsymbol{V}^{-1}$ out on the right, and identifying the expression remaining in the middle as the Taylor expansion of $e^{D}$.

**Part e) (10 points):** Use your numerical model to calculate the projected temperature change of the ocean-atmosphere system and its subcomponents. Use the nominal values of the parameters given in the table below.

| Variable | Value | Units |
|---|---|---|
| $K_a$ | 0.45 | [$Wm^{-2}K^{-1}$ yr] |
| $K_m$ | 10 | [$Wm^{-2}K^{-1}$ yr] |
| $K_0$ | 100 | [$Wm^{-2}K^{-1}$ yr] |
| $\lambda$om | 45 | [$Wm^{-2}K^{-1}$] |
| $\lambda$ma | 6 | [$Wm^{-2}K^{-1}$] |

What do your results suggest about possible changes in global temperature due to fossil fuel emissions?

**Part f) (10 points):** Use the eigenvalues of ***A*** to find the time scales (in years) for each of the model compartments. You can do this even it you choose to use `ode23b` in your model (just use MATLAB's `eig` function to do an eigen decomposition of ***A***).