1

(a) E[Time for George] $= P$(Idle Sys) $*$ (E[Time to next arrival] $+ E$[Service time]) $+ P$(Sys Busy) $* E$[Service time]

By the sum of rates

$P(\text{Busy}) = \frac{54}{60}$

$P(\text{Idle}) = 1 - \frac{54}{60}$

$E[\text{Time for George}] = (1 - \frac{54}{60}) * (\frac{1}{54} + \frac{1}{60}) + (\frac{54}{60}) * (\frac{1}{120}) = 0.661$ min

(b) Although the service time for each customer type is constant, the queueing system is no longer M/D/1 because the service time of Type 1 customers is different from that of Type 2 customers. We can view this queueing system as an M/G/1 queueing system where the service time distribution is given by the following PMF:

$$p_S(s) = \begin{cases} \frac{1}{2}, & s = 1 \text{ minute}\,, \\ \frac{1}{2}, & s = 2 \text{ minutes}\,. \end{cases}$$

The total expected cost of waiting time is given by

$$C = (\lambda_1 c_1 + \lambda_2 c_2) W_q = (\lambda_1 c_1 + \lambda_2 c_2) \frac{\lambda(1/\mu^2 + \sigma_S^2)}{2(1-\rho)}\,.$$

Since $\frac{1}{\mu^2} + \sigma_S^2 = E[S]^2 + \sigma_S^2 = E[S^2]$, C can be rewritten as

$$C = (\lambda_1 c_1 + \lambda_2 c_2) \frac{\lambda E[S^2]}{2(1-\rho)}\,.$$

We have $\mu = \frac{1}{E[S]} = 1$ customers/min $= 60$ customers/hr, $E[S^2] = \frac{1}{3600}$ hr^2, and $\rho = \frac{\lambda}{\mu} = \frac{54}{60}$. Therefore

$$C = \big(30(2 \times 60) + 24(3 \times 60)\big) \frac{54(\frac{1}{3600})}{2(1 - \frac{54}{60})} = \$594 \text{ per hour}\,.$$

(c) In this part, we have an $M/D/1$ system (a special case of the $M/G/1$), where services are deterministic and last exactly 1 minute, regardless of customer type. Let “IC” denote internal cost and "EC" denote external cost. We know that

$$\begin{aligned}
W_q &= \frac{\lambda E[S^2]}{2(1-\rho)} \\
\frac{dW_q}{d\lambda_i} &= \frac{(1-\rho)E[S_i^2] + \frac{\lambda}{\mu}E[S^2]}{2(1-\rho)^2} \\
IC(i) &= c_i W_q \\
EC(i) &= c\lambda \frac{dW_q}{d\lambda_i}
\end{aligned}$$

In this example,

$$\begin{aligned}
c_1 &= \$120 \text{ per hour} \\
c_2 &= \$180 \text{ per hour} \\
c &= \frac{30}{54}\cdot 120 + \frac{24}{54}\cdot 180 \;=\; \frac{10}{9}+\frac{12}{9} \;=\; \frac{440}{3} \\
\rho &= \frac{54}{60} = \frac{9}{10} \\
E[S^2] &= E[S_i^2] \;=\; \frac{1}{3600}\ \text{hr}^2 \\
W_q &= \frac{\frac{54}{3600}}{\frac{2}{10}} = \frac{3}{40} \\
\frac{dW_q}{d\lambda_i} &= \frac{\frac{1}{10(3600)} + \frac{9}{10(3600)}}{\frac{1}{50}} \;=\; \frac{1}{72}
\end{aligned}$$

$$\begin{aligned}
IC(1) &= 120\frac{3}{40} \;=\; \$9 \text{ per hr} \\
EC(1) &= \frac{440}{3}\cdot 54 \cdot \frac{1}{72} \;=\; \$110 \text{ per hr} \\
IC(2) &= 180\frac{3}{40} \;=\; \$13.5 \text{ per hr} \\
EC(2) &= EC(1) \;=\; \$110 \text{ per hr}
\end{aligned}$$

The external costs are the same, since $E[S_i^2] = \frac{1}{3600}$, $i \in \{1,2\}$.

(d)

$$E[George] = E[George|Busy] * P(Busy) + E[George|Idle] * P(Idle)$$

$$P(Busy) = \frac{54}{60} = \frac{9}{10}$$

$$P(Idle) = 1 - \frac{54}{60} = \frac{6}{60} = \frac{1}{10}$$

$$E[George|Busy] = \frac{1}{2} * \frac{\lambda_1}{\lambda} * \frac{1}{\mu_2} + \frac{1}{2} * \frac{\lambda_2}{\lambda} * \frac{1}{\mu_2} = 0.5min$$

$$E[George|Idle] = E[Timetonextarrival] + E[Servicetime]$$

$$E[George|Idle] = \frac{1}{54} + E[George|Busy] = 2.111min$$

$$E[George] = 0.9 * 0.5 + 0.1 * 2.111 = 0.6611min$$

Which matches the answer in Part A.

(e) Using the formula from the lecture notes:

$$C = (\lambda_1 * c_1 + \lambda_2 * c_2) * W_q$$

$$W_2 = \lambda * \frac{E[s^2}{2 * (1 - p)} = 0.0984hr$$

$$C = (30 * (2 * 60) + 24 * (3 * 60)) * (0.0984) = \$779.63$$

(f) A randomly selected customer is of type i w.p. $\frac{\lambda_i}{\lambda}$. Let $E[S^2 \mid i] = E[S_i^2]$ denote the expected squared service time, given that the customer is of type i and $P(i)$ denote the probability that the customer is type i.

$$\begin{aligned}
\rho &= \frac{30}{120} + 24 \cdot \frac{1.625}{60} = 0.9 \\
E[S^2] &= E[S^2 \mid 1]P(1) + E[S^2 \mid 2]P(2) = \left(\frac{1}{120}\right)^2 \cdot \frac{5}{9} + \left(\frac{1.625}{60}\right)^2 \frac{4}{9} = 3.6458 \text{ x } 10^{-4} \\
W_q &= \frac{\lambda E[S^2]}{2(1-\rho)} = 0.098437 \\
E[S_1^2] &= \frac{1}{14400} \\
E[S_2^2] &= 7.3351 \text{ x } 10^{-4} \\
IC(1) &= 120 W_q = \$11.8125 \text{ per hour} \\
EC(1) &= \frac{440}{3} \cdot 54 \cdot \frac{\frac{1}{10} \cdot E[S_1^2] + \frac{9}{10} \cdot E[S^2]}{0.02} = \$132.6875 \text{ per hr} \\
IC(2) &= 180 W_q = \$17.7188 \text{ per hr} \\
EC(2) &= \frac{440}{3} \cdot 54 \cdot \frac{\frac{1}{10} \cdot E[S_2^2] + \frac{9}{10} \cdot E[S^2]}{0.02} = \$158.9844 \text{ per hr}
\end{aligned}$$

In part (f), $E[S]$ is the same as in part (c). However, $E[S^2]$ and therefore $\text{Var}(S)$ have increased causing W_q and both internal costs to increase. In addition, in part (c), when $E[S_i^2] = E[S^2]$, $\frac{dW_q}{d\lambda_i} = \frac{E[S^2]}{2(1-\rho)^2}$ for both $i \in \{1, 2\}$. In contrast, in part (f) since $E[S_1^2] \neq E[S_2^2]$, the external costs are different. Both external costs increase in part (f), as compared with part (c). For the type 2 customers, it is easy to see the reason; both $E[S^2]$ and $E[S_2^2]$ have increased from their values in part (c). For the type 1 customers, $E[S_1^2]$ has decreased from its value in part (c), but the $\frac{dW_q}{d\lambda_1}$ factor in the external cost is dominated by the term involving

$E[S^2]$ rather than $E[S_1^2]$. So, the decrease in $E[S_1^2]$ is outweighed by the increase in $E[S^2]$. Lastly, the external cost due to type 2 customers is, as expected, higher than that due to type 1, since the former require a longer service time.

(g) Using the lecture notes we get:

$$f_1 = c_1 * \mu_1 = 14400$$

$$f_2 = c_2 * \mu_2 = 6642$$

Thus,we should assign type one customers priority.

(h) We use the formula

$$C = c_1 * \lambda_1 * W_1 + c_2 * \lambda_2 * W_2$$

Plugging in for the above we get:

$$C = 3600 * 0.013 + 4320 * 0.1312$$

$$C = \$614.25$$

What we learned is that the cost in the priority system is greater than that of the non-priority system, thus giving vustomers priority is more expensive.

2 To obtain the result, we have here a special case of (4.101), where $k = 2$, $\bar{L}_{qi} = 0$ $(i = 1, 2)$, and $\frac{1}{\mu_0} = \frac{1}{\mu}$. Therefore,

$$\bar{W}_q = \bar{W}_0 + \frac{1}{\mu}\bar{M}_1$$

$$\bar{W}_0 = \rho\frac{\mu E[s^2]}{2/\mu} = \frac{\lambda[\frac{1}{\mu^2} + \sigma_s^2]}{2}$$

$$\implies \bar{W}_q[1 - \rho] = \frac{\lambda[\frac{1}{\mu^2} + \sigma_s^2]}{2}$$

or

$$\bar{W}_q = \frac{\lambda[\frac{1}{\mu^2} + \sigma_s^2]}{2(1 - \rho)}$$

3

(a) We define the states (a, b, c)

Where:

a=number of planes in first stage of service

b=number of planes in second stage of service

c=number of planes in queue

From that we get the transition diagram given in the handout.

Where we note that we can split a passion process. i.e., if repairs are completed at rate μ and the probability another stage of repair is needed occurs with .6 then the transition rate from the first to the second repair stage is $\mu * .6$.

(b) We need to compute the steady state probabilities. These can be found by solving the following set of equations:

$$2P(1,0,1) + 0.8P(1,1,0) = P(0,0,0)$$

$$P(0,0,0) + 0.8p(2,1,0) + 2P(2,0,1) = 1.2p(1,1,0) + 0.6666p(1,1,0) + .8p(1,1,0)$$

$$.66666P(1,1,0) + .8P(3,1,0) + 2P(3,0,1) = 1.2P(2,1,0) + .333333P(2,1,0)$$

$$.33333P(2,1,0) = .8P(3,1,0) + 1.2P(3,1,0)$$

$$1.2P(1,1,0) = 2P(1,0,1) + .66666P(1,0,1)$$

$$.6666P(1,0,1) + 1.2P(2,1,0) = 2P(2,0,1) + .33333P(2,0,1)$$

$$.33333P(2,0,1) + 1.2P(3,1,0) = 2P(3,0,1)$$

$$P(0,0,0) + P(1,1,0) + P(2,1,0) + P(3,1,0) + P(1,0,1) + P(2,0,1) + P(3,0,1) = 1$$

Solving these we get:

$$P(0,0,0) = .4224$$

$$P(1,1,0) = .2485$$

$$P(2,1,0) = .0964$$

$$P(3,1,0) = .0161$$

$$P(1,0,1) = .1118$$

$$P(2,0,1) = .0815$$

$$P(3,0,1) = .0232$$

Thus the expected value for the number in the system is:

$$(.2485 + .1118) + 2(.0964 + .0815) + 3(.0161 + .0232) = 0.834$$

4 The airport can be modeled as an M/G/1 queueing system with a uniform service time distribution with $E[S] = 60$ sec and $\sigma_S^2 = 48$ sec^2.

(a) The yearly cost to the commercial airlines due to peak-hour delays is computed by

$$C_A = c_A \lambda_A W_q \times 1000\,,$$

where

- c_A is the average cost of 1 minute's waiting for commercial jets = \$12/min/aircraft,
- λ_A is the arrival rate of commercial jets = 40 aircraft/hr.

To compute W_q, we use the Pollaczek-Khintchine formula (4.81) for the M/G/1 queueing system.

$$W_q = \frac{\lambda(1/\mu^2 + \sigma_S^2)}{2(1-\rho)}\,,$$

where

- $\lambda = 55$ aircraft/hr $= \frac{11}{12}$ aircraft/min,
- $\mu = \frac{1}{E[S]} = 1$ aircraft/min,
- $\sigma_S^2 = 48 \text{ sec}^2 = \frac{1}{75} \text{ min}^2$,
- $\rho = \frac{\lambda}{\mu} = \frac{11}{12}$.

Note that we are using λ (not λ_A) in computing W_q. Plugging the numbers into the formula for W_q, we have $W_q = 5.57\dot{3}$ minutes. Therefore

$$C_A = \$12/\text{min/aircraft} \times 40 \text{ aircraft/hr} \times 5.57\dot{3} \text{ minutes} \times 1000 \text{ hours} = \$2,675,200\,.$$

(b) Let f be the amount of increase in landing fees. Then the arrival rate of general aviation is expressed by $15 - \frac{15}{60}f$ per hour. Since the arrival rate of commercial airlines is not affected by the increases in landing fees, the total arrival rate λ is given by

$$\lambda = \left(40 + 15 - \frac{15}{60}f\right) \text{ per hour} = \left(\frac{11}{12} - \frac{1}{240}f\right) \text{ per minute}\,.$$

The yearly cost to commercial airlines during the peak hours is now composed of the landing fees cost and the waiting (delay) cost.

$$
\begin{aligned}
C_A &= (\lambda_A f + c_A \lambda_A W_q) \times 1000 \\
&= \left(\lambda_A f + c_A \lambda_A \frac{\lambda(1/\mu^2 + \sigma_S^2)}{2(1-\rho)} \right) \times 1000 \\
&= \left(\lambda_A f + c_A \lambda_A \frac{(\frac{11}{12} - \frac{1}{240} f)(1/\mu^2 + \sigma_S^2)}{2(\frac{1}{12} + \frac{1}{240} f)} \right) \times 1000 \,.
\end{aligned}
$$

Now we want to find f that minimizes C_A.

$$
\frac{dC_A}{df} = \left(\lambda_A + c_A \lambda_A \frac{-\frac{1}{240}(1/\mu^2 + \sigma_S^2)(\frac{2}{12} + \frac{2}{240} f) - (\frac{11}{12} - \frac{1}{240} f)(1/\mu^2 + \sigma_S^2)\frac{2}{240}}{4(\frac{1}{12} + \frac{1}{240} f)^2} \right) \times 1000 = 0 \,.
$$

$$
\begin{aligned}
&\Rightarrow \quad 1 - c_A \frac{\frac{1}{120}(1/\mu^2 + \sigma_S^2)}{4(\frac{1}{12} + \frac{1}{240} f)^2} = 0 \\
&\Rightarrow \quad 4\left(\frac{1}{12} + \frac{1}{240} f \right)^2 = \frac{c_A}{120}(1/\mu^2 + \sigma_S^2) = \frac{12}{120}\left(1 + \frac{1}{75}\right) = \frac{76}{750} \\
&\Rightarrow \quad f = 18.2
\end{aligned}
$$

Hence the optimal amount of increase in landing fees is $f = \$18.2$ per aircraft.

5

(a) $(a, b)=$ a customers in the first system, b in the second

$b1=$ one server in system one is blocked other is idle

$b2=$ one server in system one is blocked other is busy

$b3=$ both servers in system one are blocked

See handout for state transition diagram

(b) Clearly $\lambda =$(# in system)P(n in system)

$\lambda = 1*$P(1 in system)+2*P(2 in system)+3*P(3 in system)

$\lambda =$(P(1,0)+P(0,1))+2*(P(2,0)+P(1,1)+P($b1$,1))+3*(P(2,1)+P($b2$,1)+P($b3$,1))