Urban OR **Fall 2004**

Problem Set 6

(Due: Wednesday, December 1, 2004)

Problem 1: Exercise 6.7 (page 442) in Larson and Odoni.

Problem 2: Problem 6.6 in L+O.

Problem 3: Problem 6.11 in L+O.

Problem 4: Problem 6.17 in L+O.

Problem 5:

Consider the Traveling Salesman Problem with Backhauls (TSPB), a version of the TSP which is as follows: Suppose we have one "station point", s, a set D of "delivery" points ($|D| = n$) and a set P of "pick-up" points ($|P| = m$). Assume the travel medium is the Euclidean plane and that all n+m+1 points in the problem are distinct, so that the Euclidean distance between any pair of points is positive. We want to design a tour of minimum length that has this description: a vehicle will begin from s, will visit first all the n points in D to deliver packages, will then (without first returning to s) visit all m points in P to pick up packages and will finally go back to s (where the tour ends) .

The following heuristic, based on the idea of the Christofides heuristic for the TSP, has been proposed recently for the TSPB (we shall leave Step 4 incomplete until the second question of our problem):

<u>Step 1:</u> Construct: (a) TD, the minimum spanning tree of D, i.e., the MST that connects all n delivery points; (b) TP, the minimum spanning tree of P, i.e., the MST that connects all m pick-up points.

Now, connect s to the delivery point in D which is closest to s. Also connect s to the pick-up point in P which is closest to s.

In this way a single tree, T, is formed which consists of TD, TP, s and the two links connecting s to TD and TP, respectively.

<u>Step 2:</u> Transform the (n+m+1)x(n+m+1) matrix of Euclidean distances between pairs of points in the problem in the following way: leave all distances between pairs of points in D unchanged, i.e., equal to the Euclidean distances between these points; similarly, leave all distances between pairs of points in P unchanged; then add a large constant K to all the other distances in the matrix. This means that if the Euclidean distance between any point $i \in D$ and any point $j \in P$ is $d(i, j)$ units, it will be changed to $d(i, j)+K$ in the transformed distance matrix; similarly, the distance between s and i in the transformed distance matrix will be $d(s, i)+K$ and the distance between s and j will be $d(s, j)+K$. Note that the same "large constant" K is added in all cases. K is chosen so that it

is much larger than any of the Euclidean distances (or sums of Euclidean distances) encountered in the problem.

Step 3: Let R be the set of odd-degree nodes in T, the tree obtained in Step 1. Find the minimum-cost pairwise matching between the nodes (points) in R, using the transformed distance matrix prepared in Step 2. Let H be the (disjoint) graph consisting of the set of links (straight lines) which correspond to this minimum-cost pairwise matching. (For example, if the points $i \in D$ and $j \in P$ are both in R and have been matched together in the minimum-cost matching, then the link (i, j) will be in H; similarly, if the points $v \in D$ and $w \in D$ are both in R and have been matched together, the link (v, w) will be in H; and so on.) Note that, in drawing the graph H, we forget about the large constant K.

Step 4: Merge H with T to obtain an Eulerian graph G (i.e., $G = T \cup H$).

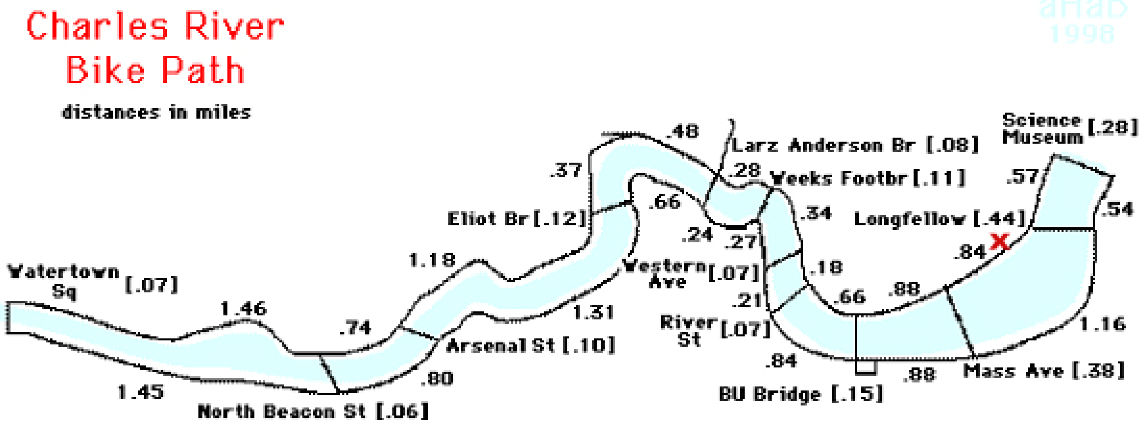
Question 1: Explain carefully but briefly why it is true that the number of (odd-degree) nodes in R which are also delivery points is an odd number (i.e., $|R \cap D|$ is an odd number). By analogy it is also true that the number of (odd-degree) nodes in R which are also pick-up points is an odd number (i.e., $|R \cap P|$ is also an odd number).

Question 2: Assume (if you have not been able to show it) that the observation in Question 1 is correct. Given this, argue carefully but briefly that:

(a) The graph G constructed in Step 4 of the algorith has an Euler tour, i.e., has no nodes of odd degree.

(b) It is possible, using only links in G, to find an Euler tour which is a feasible (not necessarily optimal) solution to the TSPB. In other words, it is possible to find on G an Euler tour that begins at s, visits first all the points in D at least once, then visits all the points in P at least once and finally returns to the origin s.

[As an aside we note that the above heuristic is the best currently available for the TSPB, in terms of worst-case performance.]



Twelve Bridges of Cambridge Problem

Problem 6 :

According to a famous Boston tourist web site, the "...Charles River Bike Path is a 16.7 mile loop along the banks of the Charles, from the Museum of Science in downtown Boston to Watertown Square and back. The dozen bridges allow for a loop walk/bike of almost any length (see map above with distances over and between bridges)." We at MIT have tabulated all of the land segment lengths and the bridge lengths, and we arrive at a total mileage of 18.27. The stated figure of 16.7 miles is in fact the length of the outer loop or cycle, including all the land paths plus the two end bridges, the Watertown Square Bridge and the Science Museum (land) bridge. The remaining 1.57 miles is the total length of the ten bridges between the two end ones. The figure shows the distances between consecutive bridges on each side of the river. The names of the twelve bridges are also shown along with the length of each bridge (in brackets, [ ]). The details are in the spreadsheet on the next page.

(a) Mr. Mike Jogger wants to run a route that covers every inch of the network of paths (on both sides of the river) and bridges at least once, but he wants to do it in minimum total distance. Create such a shortest-distance jogging path for Mike. What is the total distance he will have to run?

(b) (True Story!) Jon, a former Ph.D. student at the MIT Operations Research Center jogged every day at lunchtime. The ORC is shown with a red 'X' on the map. He numbered the 12 bridges from 1 to 12, as shown on the spreadsheet. Each day, to determine that day's jogging route, he would pick two sample values of random variables that were uniformly independently distributed on the integers 1 to 12. Those two experimental values would imply a jogging route. For instance, if he obtained the numbers 3 and 7, he would leave MIT and jog to bridge 3 and cross it, jog from bridge 3 to bridge 7 on the "Boston side" of the river, cross bridge 7, and then return back to MIT on the Cambridge side. If he picked the same two numbers, say 3 and 3, he would go to bridge 3, cross it and then immediately make a U-turn and cross it again, and then return to MIT. Carefully explain how you would determine the probability law for the random variable, "the number of miles Jon jogs each day." Please make sure to define carefully any variables or quantities you use. DO NOT work out the numerical details.

(c) Suppose Jon could move his office to any land-based (i.e., not on a bridge) location on the network, thereby freeing himself from his MIT home location.

(i) Are there other locations on the network that would result in a lower mean distance jogged each day, assuming he still selects his jogging routes randomly as described above? Can you identify one and explain why it is better than the original MIT location?

(ii) How would you think about finding an optimal location for Jon, where optimal means minimizing mean mileage jogged per day? Can you precisely formulate this problem? Will an optimal solution exist solely on the set of nodes, i.e., the juncture points between the bridges and the banks of the river?

Br. #	Bridge	length	Segments on north bank (Cambridge side)			Segments on south bank (Boston Side)		
1	Science Museum	0.28	from	to	length	from	to	length
2	Longfellow	0.44	Science Museum	Longfellow	0.57	Science Museum	Longfellow	0.54
3	Mass Ave	0.38	Longfellow	Mass Ave	0.84	Longfellow	Mass Ave	1.16
4	BU Bridge	0.15	Mass Ave	BU Bridge	0.88	Mass Ave	BU Bridge	0.88
5	River St.	0.07	BU Bridge	River st	0.66	BU Bridge	River St.	0.84
6	Western Ave	0.07	River St.	Western Ave	0.18	River St.	Western Ave	0.21
7	Weeks Footbr	0.11	Western Ave	Weeks Footbr	0.34	Western Ave	Weeks Footbr	0.27
8	Larz Anderson Br	0.08	Weeks Footbr	Larz Anderson Br	0.28	Weeks Footbr	Larz Anderson Br	0.24
9	Eliot Bridge	0.12	Larz Anderson Br	Eliot Bridge	0.85	Larz Anderson Br	Eliot Bridge	0.66
10	Arsenal St.	0.10	Eliot Bridge	Arsenal st.	1.18	Eliot Bridge	Arsenal St.	1.31
11	North Beacon St	0.06	Arsenal St.	North Beacon st	0.74	Arsenal St.	North Beacon St.	0.8
12	Watertown Sq	0.07	North Beacon St	Watertown Sq	1.46	North Beacon st	Watertown Sq	1.45
	total	1.93		total	7.98		total	8.36

Grand total = 18.27 miles

Problem 7:

Suppose we have a network G(N, A) such as the one pictured in Figure 1, which can be separated by an "isthmus edge", (s, t) into two distinct sub-networks $G(S, A_s)$ and $G(T, A_t)$ such that $S \cup T = N$ and $A_s \cup A_t \cup (s, t) = A$. (Note that the set of nodes S includes node s and the set of nodes T includes node t.) Let H(T) be the sum of the weights, h_j, of the nodes in the set T and H(S) be the sum of the weights, h_j, of the nodes in the set S.

(a) The following is known as Goldman's majority theorem": "If $H(T) \geq H(S)$ then the set of nodes T contains at least one solution to the 1-median problem on G(N, A)."

Prove the theorem. To do so, assume that the solution is at some node $y \in S$ and argue that $J(y) \geq J(t)$, which contradicts the assumption. J(.) is the objective function for the 1-median problem – see book. Note as well that t is the node on the $G(T, A_t)$ side of (s, t).

(b) Prove the following theorem: "If $H(T) \geq H(S)$ then one can find a solution to the original 1-median problem on G(N, A) by solving the 1-median problem on the sub-network $G'(T, A_t)$ which is identical to $G(T, A_t)$ except that the weight h_t of the node t (on which the edge (s, t) is incident) is replaced by $H(S) + h_t$."

To prove this statement argue as follows: We know from part (a) that the 1-median is in T. Show that for any node $y \in T$:

$$J(y) = C + [H(S) + h_t] \cdot d(y, t) + \sum_{j \in (T-t)} h_j \cdot d(y, j) \qquad (1)$$

where C is a constant and (T-t) indicates the set of nodes, T, not including the node t. Why does (1) prove our theorem?

(c) Using the theorems of parts (a) and (b) find very quickly the 1-median of the network shown in Figure 2. (For each node, an identification letter followed by the node's weight is indicated; link lengths are noted next to each link.) Note that you do not have to consider the lengths of the edges in solving this problem.

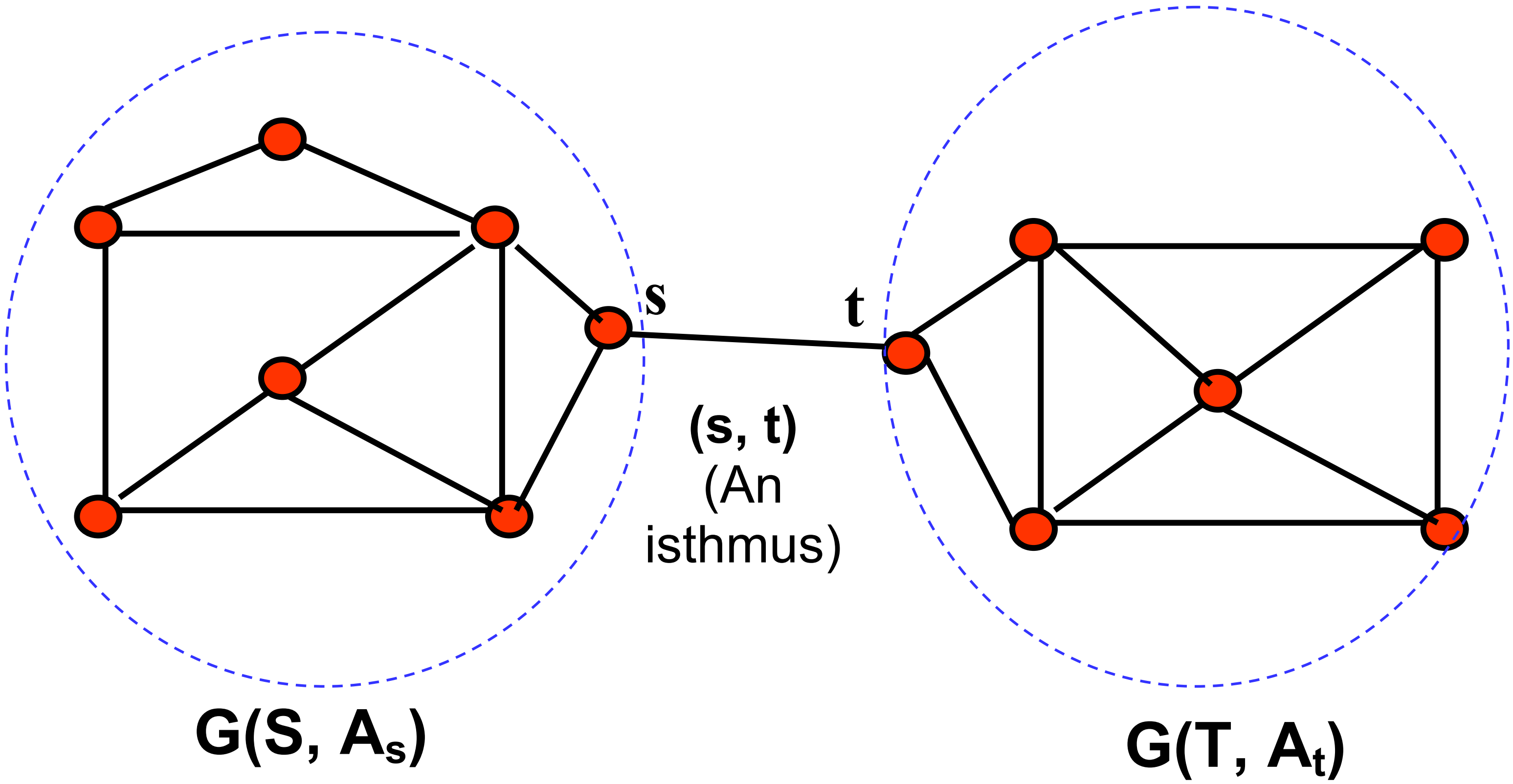
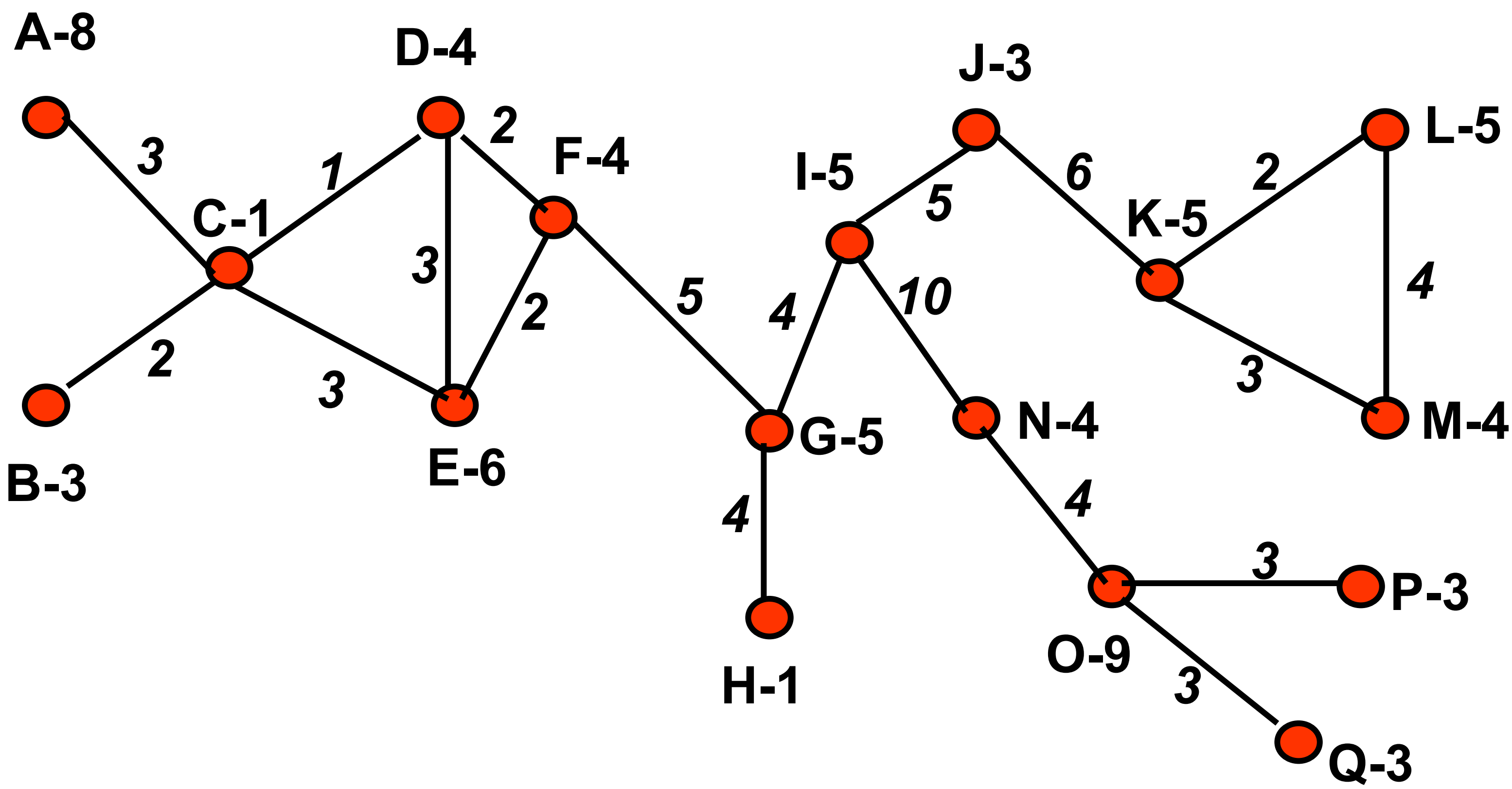


Figure 1

Figure 2


A-8
D-4
J-3
L-5
3
2
1
F-4
I-5
6
2
5
C-1
K-5
3
2
5
4
10
4
3
2
3
G-5
N-4
M-4
3
E-6
B-3
4
4
3
P-3
O-9
3
H-1
Q-3