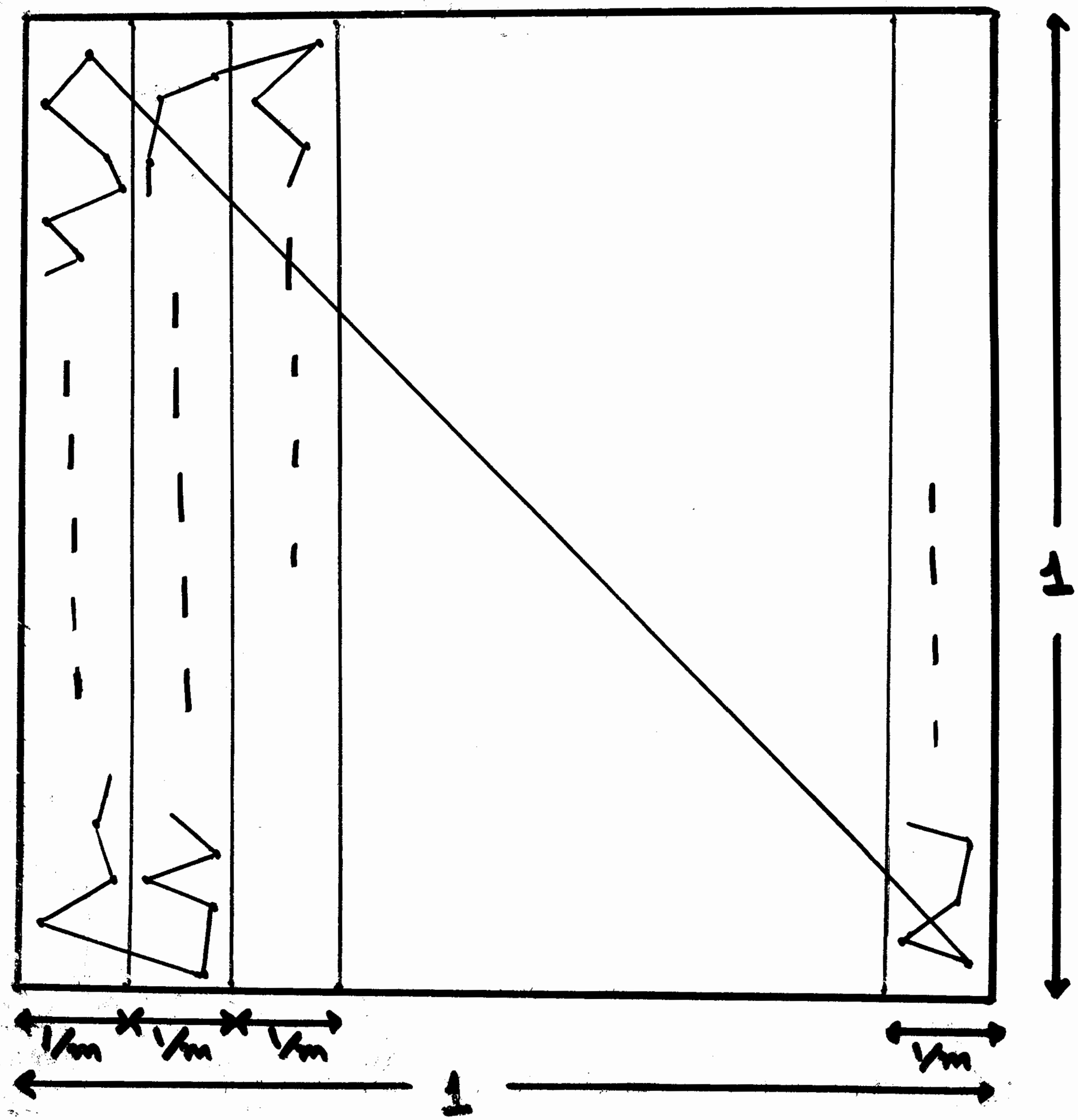
Problem 6.7
[L+O]

Construction of a tour, S_n through n points randomly and uniformly distributed in 1×1 square



$$E[L(TSP)] \leq E[L(S_n)]$$

$$\leq E\left[\sum_{i=1}^{n-1} \Delta y_i\right] + E\left[\sum_{i=1}^{n-1} \Delta x_i\right] + \sqrt{2}$$

$$\leq m + (n-m)\frac{1}{3m} + (m-1)\cdot\frac{1}{m} + \sqrt{2}$$

$$= \frac{1}{m}\left(\frac{n}{3} - 1\right) + m + \left(\sqrt{2} + \frac{2}{3}\right) \quad (*)$$

To find value of m for which RHS is minimum, set $\frac{d(RHS)}{dm} = 0$:

$$1 - \frac{1}{m^2}\left(\frac{n}{3} - 1\right) = 0 \Rightarrow m^2 = \frac{n}{3} - 1$$

$$\Rightarrow m^* = \sqrt{\frac{n}{3} - 1} \underset{n \text{ large}}{\approx} \sqrt{\frac{n}{3}}, \text{ for large } n$$

[m^* is the optimal no. of strips and it should actually be an integer]

Substituting for m^* in (*) we have that, for large n:

$$E[L(TSP)] \leq E[L(S_n)]$$
$$\leq \left(\sqrt{\frac{n}{3}} - \frac{1}{\sqrt{\frac{n}{3}}}\right) + \sqrt{\frac{n}{3}} + \left(\sqrt{2} + \frac{2}{3}\right)$$
$$\cong 2\sqrt{\frac{n}{3}} \cong 1.15\sqrt{n}$$

[Note that we could repeat this analysis for a rectangle of area $A = X_0 \cdot Y_0$:

In this case $m^* = \sqrt{\frac{X_0}{Y_0}\left(\frac{n}{3} - 1\right)} \underset{(n \text{ large})}{\cong} \sqrt{\frac{X_0}{Y_0} \cdot \frac{n}{3}}$

and

$$E[L(TSP)] \leq 2\sqrt{\frac{n}{3}} \cdot \sqrt{X_0 \cdot Y_0} \cong 1.15\sqrt{n}\sqrt{A}$$

or $$\frac{E[L(TSP)]}{\sqrt{n}\sqrt{A}} \leq 1.15.$$]

We can also derive a low bound for $E[L(TSP)]$

$$\Rightarrow \quad 0.5 \leqslant \frac{E[L(TSP)]}{\sqrt{nA}} \leqslant 1.15, \text{ as } n \to \infty$$

This suggests the following:

$$\lim_{n \to \infty} \frac{E[L(TSP)]}{\sqrt{n}\sqrt{A}} = \beta_{TSP} \qquad (**)$$

where β_{TSP} is a constant, characteristic of the TSP in a Euclidean plane. [Proved by Beardwood et al., 1959.]

Johnson (1995) claims that:

$$\beta_{TSP} = 0.7124 \pm 0.0002$$

(Significantly different from the earlier estimates of 0.75 and 0.765 cited in L+O, p.408)

EXAMPLE: 48 state capitals in Continental U.S. + Washington, DC (49 points total)

True: $L(TSP) = 10{,}070$ miles
[Rand McNally's highway distances]

Approximation: $(0.7124)\sqrt{49}\,\sqrt{3{,}022{,}400}$
$= 8{,}670$ miles (-14%)

GENERALIZATION

For several standard combinatorial problems (MST, matching, TSP, DARP, etc.) it can be shown that for uniformly, independently distributed points in a "reasonably convex and compact" region of area A in a Euclidean metric:

$$\lim_{n \to \infty} \frac{E[L(\text{optimal solution})]}{\sqrt{n}\sqrt{A}} = \text{constant}$$

[Note: Actually the results are much stronger! ⇒ "Almost surely (with prob'y 1)

$$\lim_{n \to \infty} \frac{L(\text{optimal solution})}{\sqrt{n}\sqrt{A}} = \text{constant}"$$
]

SOME OF THE CONSTANTS

$$\beta_{TSP} = 0.7124 \pm 0.0002$$

$$\beta_{MST} \cong 0.61 \quad (\beta_{MST} < \beta_{TSP})$$

$$\beta_M \cong 0.32 \quad (\text{"matching"}, \ \beta_M \leq \frac{\beta_{TSP}}{2})$$

For dial-a-ride problem in which we have n customers (i.e., $2n$ points to visit) we can easily argue that

$$\underbrace{\sqrt{2}\,\overset{0.7124}{\overset{\shortparallel}{\beta_{TSP}}}}_{\substack{\shortparallel \\ 1.01}} \leq \beta_{DARP} \leq \underbrace{2\beta_{TSP}}_{\substack{\shortparallel \\ 1.4248}}$$

$$\beta_{DARP} \cong 1.22 \quad (\text{Psaraftis, 1983})$$