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Logistical and Transportation Planning Methods
Fall 2004

Assignment 5 Solutions

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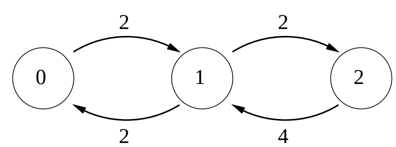
Problem 1

(a) One is tempted to say yes by setting $\rho = \frac{\lambda}{N\mu} = \frac{2}{2\times 2} = \frac{1}{2}$. But $\lambda = 2$ is not the rate at which customers are accepted into the system because we have a loss system. Thus the answer is no, and we must derive the correct figure. We can use the following aggregate birth-death process (state transition diagram for an M/M/2 queueing system with no waiting space) to compute the workloads:



The balance equations and the normalization equation are

$$2P_0 = 2P_1$$
$$2P_1 = 4P_2$$
$$P_0 + P_1 + P_2 = 1$$

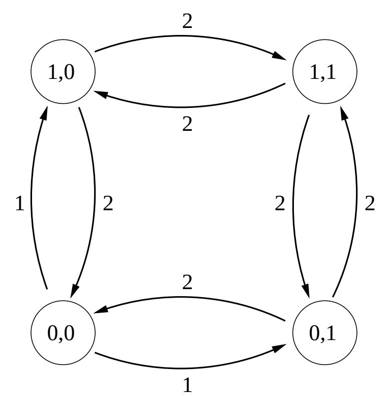
Solving the equations, we obtain

$$P_0 = \frac{2}{5}, \quad P_1 = \frac{2}{5}, \quad P_2 = \frac{1}{5}.$$

The workloads of server 1 and server 2 are then given by

$$\rho_1 = \frac{1}{2}P_1 + P_2 = \frac{2}{5}, \quad \rho_2 = \frac{1}{2}P_1 + P_2 = \frac{2}{5}\,.$$

(b) The 2-dimensional hypercube state transition diagram is given below. From the steady-state



probabilities computed in part (a) and the symmetry of the system, we have

$$P_{00} = P_0 = \frac{2}{5}, \quad P_{11} = P_2 = \frac{1}{5}, \quad P_{10} = P_{01} = \frac{1}{2}P_1 = \frac{1}{5}\,.$$

The fraction of dispatches that take server 1 to sector 2 is

$$f_{12} = \frac{\lambda_2}{(1-P_{11})\lambda}P_{10} = \frac{1}{(1-\frac{1}{5})2}\left(\frac{1}{5}\right) = \frac{1}{8}\,.$$

(c) The mean travel time to a random served customer, $\bar{T}$, is obtained by

$$\bar{T} = f_{11}\,T_1(\text{sector 1}) + f_{22}\,T_2(\text{sector 2}) + f_{12}\,T_1(\text{sector 2}) + f_{21}\,T_2(\text{sector 1})\,.$$

Since the travel speed is constant, let us first compute the mean travel distance to a random customer, $\bar{D}$.

$$\bar{D} = f_{11}\, D_1(\text{sector } 1) + f_{22}\, D_2(\text{sector } 2) + f_{12}\, D_1(\text{sector } 2) + f_{21}\, D_2(\text{sector } 1)\,.$$

Using the knowledge of Chapter 3, we have

$$D_1(\text{sector } 1) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}, \quad D_2(\text{sector } 2) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3},$$
$$D_1(\text{sector } 2) = 1 + \frac{1}{3} = \frac{4}{3}, \quad D_2(\text{sector } 1) = 1 + \frac{1}{3} = \frac{4}{3}.$$

We compute f_{11} as follows:

$$f_{11} = \frac{\lambda_1}{(1-P_{11})\lambda}(P_{00} + P_{10}) = \frac{1}{(1-\frac{1}{5})2}\left(\frac{2}{5} + \frac{1}{5}\right) = \frac{3}{8}\,.$$

Invoking the symmetries, we know

$$f_{21} = f_{12} = \frac{1}{8}, \quad f_{22} = f_{11} = \frac{3}{8}\,.$$

Putting all together,

$$\bar{D} = \frac{3}{8}\cdot\frac{2}{3} + \frac{3}{8}\cdot\frac{2}{3} + \frac{1}{8}\cdot\frac{4}{3} + \frac{1}{8}\cdot\frac{4}{3} = \frac{5}{6}\ \text{mile}\,.$$

Hence the mean travel time to a random served customer is $\bar{T} = \frac{\bar{D}}{1000}$ hr $= 3.0$ sec. This means that changes in total service time due to changes in travel time are insignificant and therefore the Markov models applies. Note that another way to compute $\bar{D}$ is

$$\bar{D} = \frac{P_{00}(\frac{2}{3}) + (P_{01} + P_{10})(\frac{1}{2} \cdot \frac{2}{3} + \frac{1}{2} \cdot \frac{4}{3})}{P_{00} + P_{01} + P_{10}} = \frac{\frac{2}{5}(\frac{2}{3}) + \frac{2}{5}(\frac{1}{2} \cdot \frac{4}{3} + \frac{1}{2} \cdot \frac{2}{3})}{\frac{4}{5}} = \frac{5}{6}\,.$$

In fact, we can obtain this form by simplifying the formula for $\bar{D}$ above. However, think about how we can obtain this directly without using the formula for $\bar{D}$ above.

(d) Consider a long time interval T. In the steady state, the average total number of customers served is $\lambda T(1 - P_{11})$. Server 1 is sent to sector 2 in the following cases:

(1) A customer arrives from sector 2, server 2 is busy, and server 1 is idle.

(2) A customer arrives from buffer zone 2, server 2 is idle outside buffer zone 2, and server 1 is idle inside buffer zone 1.

The average number of customers served by the first case is $\lambda_2 T P_{10}$. To compute the average number of customers served by the second case, let us first find the probability that server 2 is idle *outside* buffer zone 2 and server 1 is idle *inside* buffer zone 1. Using geometrical probability and the independence of the two servers, we know that the probability is $(\frac{3}{4})(\frac{1}{4})P_{00}$. Since the arrival rate from buffer zone 2 is $\frac{\lambda_2}{4}$, the average number of customers served by the second case during time interval T is $\frac{\lambda_2}{4}T(\frac{3}{4})(\frac{1}{4})P_{00}$.

Using these quantities, we obtain the fraction of dispatch assignments that send server 1 to sector 2 under the new dispatch policy as follows:

$$f'_{12} = \frac{\lambda_2 T P_{10} + \frac{\lambda_2}{4}T(\frac{3}{4})(\frac{1}{4})P_{00}}{\lambda T(1 - P_{11})} = \frac{1 \cdot \frac{1}{5} + \frac{1}{4} \cdot \frac{3}{4} \cdot \frac{1}{4} \cdot \frac{2}{5}}{2(1 - \frac{1}{5})} = \frac{35}{256} = 0.1367\,.$$

f'_{12} is greater than $f_{12} = 0.125$ as expected. Note that the state transition diagram does not change under the new dispatch policy (Why? Invoke symmetries).

(e) Let T_1 be the travel time of server 1 to a random customer and T_2 be the travel time of server 2 to a random customer. Similar to (c), the mean travel time to a random customer under the new dispatch policy is given by

$$\begin{aligned}\bar{T}' =& f'_{11}\, E[T_1 \mid \text{server 1 has been dispatched into sector 1}] + \\ & f'_{22}\, E[T_2 \mid \text{server 2 has been dispatched into sector 2}] + \\ & f'_{12}\, E[T_1 \mid \text{server 1 has been dispatched into sector 2}] + \\ & f'_{21}\, E[T_2 \mid \text{server 2 has been dispatched into sector 1}]\,.\end{aligned}$$

But the existence of buffer zones complicates matters. One way to handle this is as follows:

- Break up f'_{12} (and f'_{21}) into its two constituent parts and compute a conditional mean travel distance for each
- Do the same for f'_{11} and f'_{22}.
- Combine the results for the final answer.

The numerical value is less than that of part (c), because we tend to dispatch the closer available server (not always successful, though).

Although it is not required in the question, let us compute $\bar{T}'$ exactly. We define the following events:

- CB: A customer is in a buffer zone.

- SAB: Server of the adjacent sector is in its buffer zone.
- SHB: Server of home sector is in its buffer zone.

Let us denote by CB^c the complement event of CB, which means that a customer is not in a buffer zone. Other complement events are defined similarly. Then in the state where both servers are available, with probability P_{00}, we have eight mutually exclusive, collective exhaustive events: $(\text{CB}\cap\text{SAB}\cap\text{SHB})$, $(\text{CB}^c\cap\text{SAB}\cap\text{SHB})$, $(\text{CB}\cap\text{SAB}^c\cap\text{SHB})$, $(\text{CB}\cap\text{SAB}\cap\text{SHB}^c)$, $(\text{CB}^c\cap\text{SAB}^c\cap\text{SHB})$, $(\text{CB}^c\cap\text{SAB}\cap\text{SHB}^c)$, $(\text{CB}\cap\text{SAB}^c\cap\text{SHB}^c)$, and $(\text{CB}^c\cap\text{SAB}^c\cap\text{SHB}^c)$.

Let us abbreviate these events in binary, for example, $(\text{CB}\cap\text{SAB}\cap\text{SHB}) = (111)$, $(\text{CB}^c\cap\text{SAB}\cap\text{SHB}^c) = (010)$, etc. Then we can write, using the techniques from Chapter 3 for the conditional mean travel distances,

$$\bar{D}' = \frac{P_{00}A + (P_{01}+P_{10})(\frac{1}{2}\cdot\frac{2}{3}+\frac{1}{2}\cdot\frac{4}{3})}{P_{00}+P_{01}+P_{10}},$$

where A is

$$A = \left(\frac{1}{3}+\frac{1}{4}\right)P(110) + \left[\frac{1}{3}+\left(\frac{1}{2}\cdot\frac{3}{4}+\frac{1}{2}\cdot\frac{1}{4}\right)\right]P(100)+$$
$$\left(\frac{1}{3}+\frac{1}{3}\cdot\frac{1}{4}\right)\big(P(101)+P(111)\big) + \left(\frac{1}{3}+\frac{1}{3}\cdot\frac{3}{4}\right)\big(P(010)+P(000)\big)+$$
$$\left[\frac{1}{3}+\left(\frac{1}{2}\cdot\frac{1}{4}+\frac{1}{2}\cdot\frac{3}{4}\right)\right]\big(P(001)+P(011)\big).$$

We have

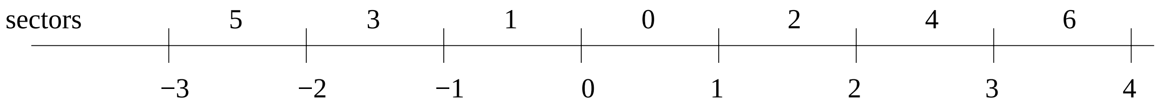


Figure 1: Sector Layout

$$P(110) = \frac{1}{4}\cdot\frac{1}{4}\cdot\frac{3}{4}, \quad P(100) = \frac{1}{4}\cdot\frac{3}{4}\cdot\frac{3}{4}, \quad P(101) = \frac{1}{4}\cdot\frac{3}{4}\cdot\frac{1}{4}, \quad P(111) = \frac{1}{4}\cdot\frac{1}{4}\cdot\frac{1}{4},$$

$$P(010) = \frac{3}{4}\cdot\frac{1}{4}\cdot\frac{3}{4}, \quad P(000) = \frac{3}{4}\cdot\frac{3}{4}\cdot\frac{3}{4}, \quad P(001) = \frac{3}{4}\cdot\frac{3}{4}\cdot\frac{1}{4}, \quad P(011) = \frac{3}{4}\cdot\frac{1}{4}\cdot\frac{1}{4}.$$

Plugging all numbers, we obtain $\bar{D}' = \frac{1271}{1536} = 0.82747 < \bar{D} = \frac{5}{6} = 0.83333$. The mean travel time to a random customer is $\bar{T}' = \frac{\bar{D}'}{1000} = 2.9789$ sec $< \bar{T} = 3$ sec. So, we do get an expected improvement in mean response distance (time), but not a large one. The fact that we have more inter-sector dispatches *does not* necessarily mean that mean response distance (time) will increase.

(f) First, do not use Carter, Chaiken, and Ignall formula (Equation (5.18)). It only applies when server locations are fixed. The best option is to compute $\bar{T}(x)$, where x is the location of a boundary line, and use calculus to find an optimal value of x (as we did in the 2-server numerical example in the book and in class). The problem with Equation (5.18) is that $T_1(B)$ and $T_2(B)$ depend on the location of the boundary line separating sectors 1 and 2. This is because each available server patrols uniformly its sector while it is idle and thus its travel time in B depends on sector design.

Problem 2

(a)(i) Figure 1 gives the layout of the sectors. The SCM strategy determines which unit should be dispatched by assuming that the incident and units are located at the centers of mass of their respective sectors. That is, the dispatching strategy assumes that the incident occurs at fixed location $\frac{1}{2}$ and that unit i is located at $\frac{i+1}{2}$ for i even and at $-\frac{i}{2}$ for i odd. Under SCM, it is obvious that the dispatch strategy will operate according to the following fixed list of preferences: 0, 1 or 2, 3 or 4, etc. That is, dispatch unit 0 if it is available. Otherwise, if one of units 1 or 2 is available, dispatch it (or either one if both are available). Otherwise, if one of units 3 or 4 is available, dispatch one of them. Etc.

For $i = 0$, let S_i denote the event that unit 0 is available. For $i \geq 1$, let S_i denote the event that units $0, \ldots, 2(i-1)$ are unavailable, but at least one of units $2i$ or $2i-1$ is available. We know that

$$\begin{aligned}
P(S_0) &= 1 - \rho \\
P(S_1) &= \rho(1 - \rho^2) \\
P(S_2) &= \rho^3(1 - \rho^2) \\
P(S_3) &= \rho^5(1 - \rho^2) \\
\vdots &= \vdots \\
P(S_i) &= \rho^{2i-1}(1 - \rho^2), \ i \geq 1
\end{aligned}$$

We want to compute the expected actual distance traveled using the SCM dispatching policy. Note that we treat the incidents and units as having deterministic locations only in setting the dispatch preferences (and computing the resulting probabilities that particular units are dispatched) and not in computing the expected distance given a particular dispatch choice. Using the Total Expectation

Theorem,

$$\begin{aligned}
\bar{D}_1(\rho) &= \sum_{i=0}^{\infty} E[D \mid S_i]P(S_i) \\
&= (1-\rho)E[D \mid S_0] + \sum_{i=1}^{\infty} E[D \mid S_i]P(S_i) \\
E[D \mid S_0] &= E[\mid X - Y \mid], \text{ where } X, Y \sim \mathrm{U}(0,1) \\
&= \frac{1}{3} \\
i \geq 1,\ E[D \mid S_i] &= E[X - Y], \text{ where } X \sim \mathrm{U}(i, i+1), Y \sim \mathrm{U}(0,1) \\
&= \frac{i+i+1}{2} - \frac{1}{2} \ = \ i \\
\bar{D}_1(\rho) &= \frac{1}{3}(1-\rho) + \sum_{i=1}^{\infty} i\rho^{2i-1}(1-\rho^2) \\
&= \frac{1}{3}(1-\rho) + \rho\sum_{i=0}^{\infty} i\left(\rho^2\right)^{i-1}(1-\rho^2) \\
&= \frac{1}{3}(1-\rho) + \frac{\rho}{1-\rho^2}
\end{aligned}$$

where the last equality follows since $\sum_{i=0}^{\infty} i\left(\rho^2\right)^{i-1}(1-\rho^2)$ is the expected value of a geometric RV with probability of success equal to $1-\rho^2$.

(a)(ii) Under the MCM strategy, the dispatcher uses the incident's exact location in determining which unit to dispatch. In particular, the dispatcher takes as an input the fixed (nonrandom) $x \in [0,1]$ giving the incident's location and chooses unit i among the available units that minimizes $E[|X_i - x|]$ for this value of x. The expectation is taken over the location of the unit i, given by X_i.

Without loss of generality, we can assume that $x \in \left[0, \frac{1}{2}\right]$, where x gives a particular experimental value of the location of the incident. That is, we can assume that x is in the half of sector 0 closest to sector 1. If $x \in \left(\frac{1}{2}, 1\right]$, then we could renumber the sectors by swapping the numbering of sectors 1 and 2, 3 and 4, 5 and 6, etc. s.t. x is in the half of sector 0 closest to sector 1 under the new labeling scheme. Note that $E[|X_0 - x|] = \int_0^x (x-u)du + \int_x^1 (u-x)du = \frac{1}{2} - x + x^2 \leq \frac{1}{2}\ \forall x \in \left[0, \frac{1}{2}\right]$. $E[|X_1 - x|] = \frac{1}{2} + x$. Using this fact and by inspection, $\forall x \in \left[0, \frac{1}{2}\right]$, $E[|X_i - x|] < E[|X_{i+1} - x|]$ $i \in \{0, 1, \dots\}$. So, we dispatch unit 0 if available. Otherwise, we dispatch unit 1 if available. If not, we try unit 2, etc. Let S_0 denote the event that unit 0 is available. For $i \geq 1$, let S_i denote the event that unit i is available but units $0, \dots, i-1$ are unavailable. $P(S_i) = \rho^i(1-\rho)$. (The S_i events defined here are different from those in part (i), with the exception of S_0). Under case S_i, unit i is dispatched.

In each of the mutually exclusive cases $S_0, S_1, S_2, \dots$, we must determine the expected travel distance between the dispatched unit and the incident. For this purpose, we treat X, the location of the incident, as a RV. Since we restricted (without loss of generality) X to lie in $\left[0, \frac{1}{2}\right]$, we can say that $X \sim \mathrm{U}\left(0, \frac{1}{2}\right)$.

$$\begin{aligned}
\bar{D}_2(\rho) &= \sum_{i=0}^{\infty} E[D \mid S_i] P(S_i) \\
E[D \mid S_i] &= E[|X_i - X|] \\
E[|X_i - X|] &= E[X - X_i] \ = \ \frac{1}{4} + \frac{i}{2},\ i \in \{1, 3, 5, \dots\} \\
E[|X_i - X|] &= E[X_i - X] \ = \ \frac{3}{4} + \frac{i-1}{2},\ i \in \{2, 4, 6, \dots\} \\
E[D|S_i] &= \begin{cases} \frac{1}{3}, & i = 0 \\ \frac{1}{4}[2i+1], & i \in \{1, 2, 3, \dots\} \end{cases} \\
\bar{D}_2(\rho) &= \frac{1}{3}(1-\rho) + \sum_{i=1}^{\infty} \frac{2i+1}{4} \rho^i (1-\rho) \\
&= \frac{1}{3}(1-\rho) + \frac{\rho}{2} \sum_{i=1}^{\infty} i\rho^{i-1}(1-\rho) + \frac{\rho}{4} \sum_{i=1}^{\infty} \rho^{i-1}(1-\rho)
\end{aligned}$$

We can simplify the two summations by noting that the first is equivalent to the expected value of a geometric RV with probability of success in any given trial $1-\rho$, and that the second summation is the sum of all disjoint outcomes for this geometric RV. Therefore

$$\begin{aligned}\bar{D}_2(\rho) &= \frac{1}{3}(1-\rho) + \frac{\rho}{2(1-\rho)} + \frac{\rho}{4} \\ &= \frac{1}{3} - \frac{1}{12}\rho + \frac{\rho}{2(1-\rho)}\end{aligned}$$

(a)(iii) In the MCM strategy, the dispatcher used only the exact location of the incident. Under the "Closest Car" dispatch strategy, the dispatcher knows and uses the exact locations of the incident *and* all available units. As the name suggests, under this strategy, the available unit closest to the incident is dispatched. As in part (ii), we can assume without loss of generality that $x \in \left[0, \frac{1}{2}\right]$ (otherwise, simply renumber the intervals so this property holds).

To give some motivation for the relevant events on which we need to condition, consider the following. Under the restriction that the incident occurs in $\left[0, \frac{1}{2}\right]$, if both units 0 and 1 are available, then one of the two will be the closest available unit to the incident. If unit 0 is available and 1 is not, then unit 0 is obviously dispatched, since, even if unit 2 is available, unit 0 is always closer to the incident than unit 2. Consider the case where unit 0 is unavailable but units 1 and 2 are available. The closest unit will obviously be either 1 or 2. However, one can imagine values of $x \in \left[0, \frac{1}{2}\right]$ for which unit 1 is closest and other values for which unit 2 is closest. We are now ready to define the general set of events on which we want to condition.

- S_i^2 will denote the event that units $0, \ldots, i-1$ are unavailable and units i and $i+1$ are available. $P(S_i^2) = \rho^i(1-\rho)^2$.

- S_i^1 will denote the event that units $0, \dots, i-1, i+1$ are unavailable and unit i is available.

  $P(S_i^1) = \rho^{i+1}(1-\rho)$.

When considering $i \in \{0, 1, 2, \dots\}$, these events are mutually exclusive and exhaust the sample space. It is easy to see that, in case S_i^1, unit i will always be the closest unit and therefore the dispatched unit, regardless of the exact value of $x \in \left[0, \frac{1}{2}\right]$. Moreover,

$$E[D \mid S_i^1] = \begin{cases} \frac{1}{3}, & i = 0 \\ \frac{2i+1}{4}, & i \in \{1, 2, \dots\} \end{cases}$$

It is much more difficult to compute $E[D \mid S_i^2]$. To do so, we will use the "never fail" method of computing the CDF of D given S_i^2. In particular, we only need to compute the CDFs of D given S_0^2 and S_1^2. We will then be able to extend these results to compute $E[D \mid S_i^2]$ for $i \in \{1, 2, \dots\}$. Actually, it is much easier to compute $P(D > d \mid S_0^2)$ rather than $P(D \leq d \mid S_0^2)$ since the former is the probability that neither unit 0 or 1 is within d units of the incident. It is obvious that $P(D > d \mid S_0^2) = 1$ for $d \leq 0$ and $P(D > d \mid S_0^2) = 0$ for $d \geq 1$. For $d \in [0, 1]$, we will compute this function by conditioning on the location of the incident. As in part (ii), let X be the RV giving the location of the incident. $X \sim \mathrm{U}\left(0, \frac{1}{2}\right)$. So,

$$f_X(x) = \begin{cases} 2, & x \in \left[0, \frac{1}{2}\right] \\ 0, & \text{otherwise} \end{cases}$$

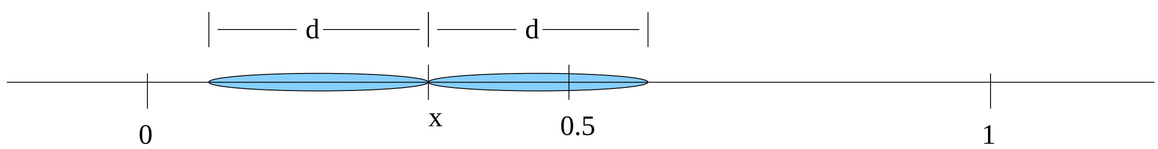


Figure 2: Sector Diagram for "Closest Car" and $d \in \left[0, \frac{1}{2}\right]$

We wish to compute

$$\begin{aligned} P(D > d \mid S_0^2, X = x) &= P(x - X_1 > d, |X_0 - x| > d) \\ &= P(x - X_1 > d) P(|X_0 - x| > d), \text{ by independence of } X_0, X_1 \end{aligned}$$

The first factor gives the probability that unit 1 is more than d units away from the incident located at x. The second factor gives the probability that unit 0 is more than d units away from the incident. Since we are working with uniform distributions, these probabilities can be found by considering the percentage of sectors 1 and 0, respectively, not covered by the interval $[x-d, x+d]$. Since the sectors have length 1, these percentages are simply given by the total length within the sector not covered by $[x - d, x + d]$.

From Figure 1, it is clear we must analyze this probability in several cases. First, if $d \in \left[0, \frac{1}{2}\right]$, there are some values of x for which X_1 is never within d units of x (namely when $d - x < 0$. For such values of x and d, X_0 can be more than d units away from x by being either to the left of $x - d$ or to the right of $x + d$. See Figure 2 for illustration. For $d \in \left(\frac{1}{2}, 1\right]$, the situation is different. X_1 is not always more than d units away from x. Furthermore, X_0 can be more than d units away from x only by being to the right of $x + d$. If $x + d > 1$, X_0 is always within d units of x. See Figure 3 for illustration.

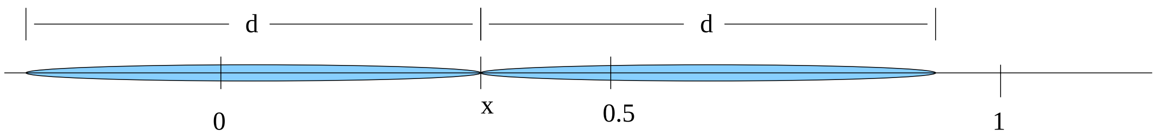


Figure 3: Sector Diagram for "Closest Car" and $d \in \left(\frac{1}{2}, 1\right]$

Case	Subcase	$P(\lvert X_0 - x\rvert > d)$	$P(x - X_1 > d)$	$P(D > d \mid S_0^2, X = x)$
$d \in \left[0, \frac{1}{2}\right]$	$d < x$	$(x - d) + (1 - (x + d))$	1	$1 - 2d$
$d \in \left[0, \frac{1}{2}\right]$	$d \geq x$	$1 - (x + d)$	$1 - (d - x)$	$(1 - (x + d))(1 - (d - x))$
$d \in \left(\frac{1}{2}, 1\right]$	$x + d > 1$	0	who cares	0
$d \in \left(\frac{1}{2}, 1\right]$	$x + d \leq 1$	$1 - (x + d)$	$1 - (d - x)$	$(1 - (x + d))(1 - (d - x))$

So, for $d \in \left[0, \frac{1}{2}\right]$,

$$\begin{aligned}
P(D > d \mid S_0^2) &= \int_0^{\frac{1}{2}} P(D > d \mid S_0^2, X = x) f_X(x) dx \\
&= 2\int_0^d (1 - (d - x))(1 - (d + x))\, dx + 2\int_d^{\frac{1}{2}} (1 - 2d) dx \\
&= \frac{4}{3}d^3 + 1 - 2d \\
P(D \leq d \mid S_0^2) &= 2d - \frac{4}{3}d^3 \\
f_{D \mid S_0^2}(d) &= 2 - 4d^2
\end{aligned}$$

For $d \in \left(\frac{1}{2}, 1\right]$, we have

$$\begin{aligned}
P(D > d \mid S_0^2) &= \int_0^{\frac{1}{2}} P(D > d \mid S_0^2, X = x) f_X(x) dx \\
&= 2 \int_0^{1-d} \left(1 - (d - x)\right)\left(1 - (x + d)\right) dx \\
&= \frac{4}{3} - 4d + 4d^2 - \frac{4}{3} d^3 \\
P(D \leq d \mid S_0^2) &= -\frac{1}{3} + 4d - 4d^2 + \frac{4}{3} d^3 \\
f_{D|S_0^2}(d) &= 4 - 8d + 4d^2
\end{aligned}$$

Therefore,

$$\begin{aligned}
E[D \mid S_0^2] &= \int_0^1 u f_{D|S_0^2}(u) du \\
&= \int_0^{\frac{1}{2}} u \left(2 - 4u^2\right) du + \int_{\frac{1}{2}}^1 u \left(4 - 8u + 4u^2\right) du \\
&= \frac{7}{24}
\end{aligned}$$

Using the same kind of reasoning, we can compute $E[D \mid S_1^2]$. First, let us derive $P(D > d \mid S_1^2)$. It is clear that $P(D > d \mid S_1^2) = 1$ for $d \leq 0$ and $P(D > d \mid S_1^2) = 0$ for $d \geq \frac{3}{2}$. Again, we will condition on $X = x$.

$$\begin{aligned}
P(D > d \mid S_1^2, X = x) &= P(x - X_1 > d, X_2 - x > d) \\
&= P(x - X_1 > d) P(X_2 - x > d), \text{ by independence of } X_1, X_2
\end{aligned}$$

The first factor gives the probability that unit 1 is more than d units away from the incident. The second factor gives the analogous probability for unit 2. As before, we must break our analysis into cases. For $d \in \left[0, \frac{1}{2}\right]$, when $d < x$, X_1 and X_2 are always more than d units away from x. In contrast, for $d \in \left(\frac{1}{2}, 1\right]$, X_1 is not always more than d units away from x. However, for $x + d < 1$, X_2 is always more than d units away from x. For $d \in \left(1, \frac{3}{2}\right]$, if d is too large (i.e. $d - x > 1$), then X_1 is always within d of x.

Case	Subcase	$P(x - X_1 > d)$	$P(X_2 - x > d)$	$P(D > d \mid S_0^2, X = x)$
$d \in \left[0, \frac{1}{2}\right]$	$d < x$	1	1	1
$d \in \left[0, \frac{1}{2}\right]$	$d \geq x$	$1 - (d - x)$	1	$1 - (d - x)$
$d \in \left(\frac{1}{2}, 1\right]$	$x + d < 1$	$1 - (d - x)$	1	$1 - (d - x)$
$d \in \left(\frac{1}{2}, 1\right]$	$x + d \geq 1$	$1 - (d - x)$	$2 - (x + d)$	$(1 - (d - x))(2 - (x + d))$
$d \in \left(1, \frac{3}{2}\right]$	$d - x \leq 1$	$1 - (d - x)$	$2 - (x + d)$	$(1 - (d - x))(2 - (x + d))$
$d \in \left(1, \frac{3}{2}\right]$	$d - x > 1$	0	who cares	0

So, for $d \in \left[0, \frac{1}{2}\right]$, we have that

$$
\begin{aligned}
P(D > d \mid S_1^2) &= \int_0^{\frac{1}{2}} P(D > d \mid S_1^2, X = x) f_X(x) dx \\
&= 2 \int_0^d (1 - (d - x))\, dx + 2 \int_d^{\frac{1}{2}} dx \\
&= 1 - d^2 \\
P(D \leq d \mid S_1^2) &= d^2 \\
f_{D|S_1^2}(d) &= 2d
\end{aligned}
$$

So, for $d \in \left(\frac{1}{2}, 1\right]$, we have

$$\begin{aligned}
P(D > d \mid S_1^2) &= \int_0^{\frac{1}{2}} P(D > d \mid S_1^2, X = x) f_X(x) dx \\
&= 2\int_0^{1-d} (1-(d-x))\,dx + 2\int_{1-d}^{\frac{1}{2}} (1-(d-x))\,(2-(x+d))\,dx \\
&= \frac{5}{6} + d - 3d^2 + \frac{4}{3}d^3 \\
P(D \le d \mid S_1^2) &= \frac{1}{6} - d + 3d^2 - \frac{4}{3}d^3 \\
f_{D|S_1^2}(d) &= -1 + 6d - 4d^2
\end{aligned}$$

Finally, for $d \in \left(1, \frac{3}{2}\right]$, we obtain

$$\begin{aligned}
P(D > d \mid S_1^2) &= \int_0^{\frac{1}{2}} P(D > d \mid S_1^2, X = x) f_X(x) dx \\
&= 2\int_{d-1}^{\frac{1}{2}} (1-(d-x))\,(2-(x+d))\,dx \\
&= \frac{9}{2} - 9d + 6d^2 - \frac{4}{3}d^3 \\
P(D \le d \mid S_1^2) &= -\frac{7}{2} + 9d - 6d^2 + \frac{4}{3}d^3 \\
f_{D|S_1^2}(d) &= 9 - 12d + 4d^2
\end{aligned}$$

Now, we can compute

$$\begin{aligned}
E[D \mid S_1^2] &= \int_0^{\frac{3}{2}} u f_{D|S_1^2}(u) du \\
&= \int_0^{\frac{1}{2}} u\,(2u)\,du + \int_{\frac{1}{2}}^{1} u\left(-1 + 6u - 4u^2\right) du \\
&\quad + \int_1^{\frac{3}{2}} u\left(9 - 12u + 4u^2\right) du \\
&= \frac{17}{24}
\end{aligned}$$

From the layout of the sectors as shown in Figure 1, it is clear that for $i \in \{2, 3, \dots\}$, we simply add $\frac{i-1}{2}$ to $E[D \mid S_1^2]$ to derive $E[D \mid S_i^2]$. Therefore

$$E[D \mid S_i^2] = \begin{cases} \frac{7}{24}, & i = 0 \\ \\ \frac{17}{24} + \frac{i-1}{2}, & i \in \{1, 2, \dots\} \end{cases}$$

Using the total expectation theorem

$$\begin{aligned} \bar{D}_3(\rho) &= \sum_{i=0}^{\infty} \left(E[D \mid S_i^1]P(S_i^1) + E[D \mid S_i^2]P(S_i^2) \right) \\ &= \frac{1}{3}\rho(1-\rho) + \frac{7}{24}(1-\rho)^2 + \sum_{i=1}^{\infty} \left(\frac{2i+1}{4} \right) \rho^{i+1}(1-\rho) + \sum_{i=1}^{\infty} \left(\frac{17}{24} + \frac{i-1}{2} \right) \rho^i (1-\rho)^2 \\ \dots &\quad \text{after some algebraic manipulation...} \\ &= \frac{7}{24} + \frac{11}{24}\rho + \frac{\rho^2}{2(1-\rho)} \end{aligned}$$

(b)

$$\begin{aligned} \epsilon_{12}(\rho) &= \frac{1}{3}(1-\rho) + \frac{\rho}{1-\rho^2} - \frac{1}{3} + \frac{1}{12}\rho - \frac{\rho}{2(1-\rho)} \\ &= -\frac{\rho}{4} + \frac{2\rho - \rho - \rho^2}{2(1-\rho^2)} \\ &= -\frac{\rho}{4} + \frac{\rho}{2(1+\rho)} \\ &= \frac{2\rho - \rho - \rho^2}{4(1+\rho)} = \rho \frac{1-\rho}{4(1+\rho)} \end{aligned}$$

$$
\begin{aligned}
\epsilon_{13}(\rho) &= \frac{1}{3}(1-\rho) + \frac{\rho}{1-\rho^2} - \frac{7}{24} - \frac{11}{24}\rho - \frac{\rho^2}{2(1-\rho)} \\
&= \frac{1}{24} - \frac{19}{24}\rho + \frac{2\rho - \rho^2 - \rho^3}{2(1-\rho^2)} \\
&= \frac{1 - \rho^2 - 19\rho(1-\rho^2) + 24\rho - 12\rho^2 - 12\rho^3}{24(1-\rho^2)} \\
&= \frac{1 + 5\rho - 13\rho^2 + 7\rho^3}{24(1-\rho^2)} \\
&= \frac{(1-\rho)(1 + 6\rho - 7\rho^2)}{24(1-\rho^2)} \\
&= \frac{1 + 6\rho - 7\rho^2}{24(1+\rho)}
\end{aligned}
$$

$$
\begin{aligned}
\epsilon_{23}(\rho) &= \frac{1}{3} - \frac{1}{12}\rho + \frac{\rho}{2(1-\rho)} - \frac{7}{24} - \frac{11}{24}\rho - \frac{\rho^2}{2(1-\rho)} \\
&= \frac{1}{24} - \frac{13}{24}\rho + \frac{\rho}{2} \\
&= \frac{1}{24}(1-\rho)
\end{aligned}
$$

Graphs of these functions are shown in Figure 4. For $\rho \to 0$, $\epsilon_{12}(\rho) = 0$ and ϵ_{ij} is close to zero for the other two pairs of i and j. This indicates that, when most units tend to be free, the expected travel distances are nearly the same under all dispatch strategies for these limiting values of ρ. This result makes intuitive sense. For ρ near 0, unit 0 (the unit in the same sector as the incident) is usually dispatched, regardless of the dispatch strategy. However, under the "closest car" dispatch strategy, it is sometimes advantageous to dispatch unit 1 or 2 over unit 0 (hence the savings of $\frac{1}{24}$). For $\rho \to 1$, $\epsilon_{ij}(\rho) = 0$, $\forall i, j$. In this second limiting case, we expect very few of the units to be available at the same time. Accordingly, we won't have the opportunity to be picky

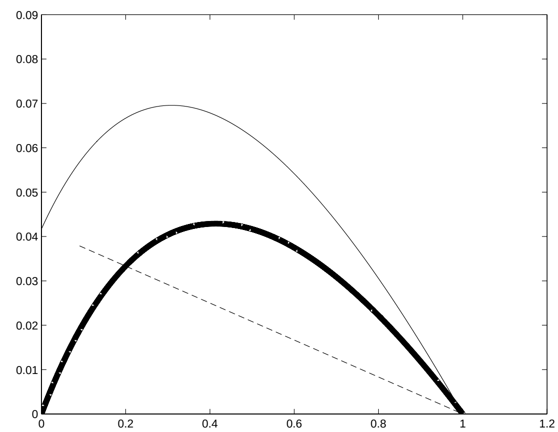


Figure 4: $\epsilon_{12}(\rho)$, $\epsilon_{13}(\rho)$, $\epsilon_{23}(\rho)$

about which car to dispatch, since "beggars can't be choosers." As a result, the dispatch strategy should not make a difference in expected travel distance.

For intermediate values of ρ, as ρ increases, the probability of out-of-sector dispatching increases. For such intermediate values, the "closest car" dispatch strategy performs best, followed by MCM, and then SCM. That ϵ_{12} and ϵ_{13} have single maxima is explained by the fact that, as ρ increases, at first, the more precise dispatch strategy works better. Once ρ becomes too large, we reach a situation in which this added level of pickiness does not make a difference since very few cars are available. Lastly, as we would expect, the "closest car" strategy is most powerful when we have a lot of choice – i.e. when ρ is low and therefore many of the units are simultaneously available.

Problem 3

(a) Let us denote the state of the system by a 3-character list of the status of each server (0 indicates free and 1 indicates busy), where the status of server 1 is given by the right-most character and that of server 3 is given by the left-most character. Let P_n denote the probability that a total of n emergencies are in the system. By the symmetry of the problem, it is clear that

$$\begin{aligned} P_{100} = P_{010} = P_{001} &= \frac{1}{3}P_1 \\ P_{110} = P_{101} = P_{011} &= \frac{1}{3}P_2 \end{aligned}$$

Since all mean service times are equal, P_n can be found by looking at the equivalent $M/M/3$ system with $\lambda = 2$ and $\mu = 1$. (See pp. 305-07 of the textbook for a discussion). From equations (4.46) and (4.44) in the Urban OR textbook, we obtain that

$$\begin{aligned} P_0 &= \frac{1}{1+2+2+\frac{4}{3(1-\frac{2}{3})}} \\ &= \frac{1}{1+2+2+4} = \frac{1}{9} \\ P_1 = P_2 &= \frac{2}{9} \\ P_n &= \frac{2^n}{54 \cdot 3^{n-3}},\ n \in \{3,4,5,\dots\} \end{aligned}$$

Furthermore, by the definition of the states, $i \in \{1,2,3\}$, we have

$$\begin{aligned} \rho_i &= \frac{1}{3}P_1 + \frac{2}{3}P_2 + \sum_{j=3}^{\infty} P_j \\ &= \frac{2}{27} + \frac{4}{27} + (1 - P_0 - P_1 - P_2) \\ &= \frac{2}{3} \end{aligned}$$

(b) Without loss of generality (by the symmetry of the problem), we can analyze the mean travel time by assuming that the call originates from district 1. We will use the Total Expectation Theorem by conditioning on the following mutually exclusive and exhaustive events

- A: unit 1 is available when the call arrives. $P(A) = 1 - \rho_1 = \frac{1}{3}$.
- B: unit 1 is unavailable when the call arrives but at least one of units 2 or 3 is available. $P(B) = P_{001} + P_{101} + P_{011} = \frac{2}{9}$
- C: no unit is available when the call arrives. $P(C) = 1 - P_0 - P_1 - P_2 = \frac{4}{9}$.

Recall that if unit 1 is unavailable but units 2 and 3 are available, then one of 2 or 3 is dispatched to the call. In either case, the expected travel time is 2 minutes. Now consider case C. Due to the memorylessness of the exponential distribution and the fact that the service time distributions are the same for each unit, the call is equally likely to be served by any of the three units. So, the expected travel time in this case is $\frac{1}{3}\left(\frac{1}{2} + 2 + 2\right) = \frac{3}{2}$. Putting this all together, we have that

$$
\begin{aligned}
E[T] &= E[T \mid A]P(A) + E[T \mid B]P(B) + E[T \mid C]P(C) \\
&= \frac{1}{2} \cdot \frac{1}{3} + 2\frac{2}{9} + \frac{3}{2} \cdot \frac{4}{9} \\
&= \frac{23}{18} \approx 1.28 \text{ minutes}
\end{aligned}
$$

(c) Again, we can condition on events A, B, and C. For event A, $T \sim \mathrm{U}(0,1)$. For event B, $T \sim \mathrm{U}(1,3)$. For event C, $T \sim \mathrm{U}(0,1)$ w.p. $\frac{1}{3}$ (unit 1 becomes free before 2 or 3) and $T \sim \mathrm{U}(1,3)$

w.p. $\frac{2}{3}$ (one of units 2 or 3 becomes free before unit 1). That is

$$\begin{aligned}
f_{T|A}(t) &= \begin{cases} 1, & t \in [0,1] \\ 0, & \text{otherwise} \end{cases} \\
f_{T|B}(t) &= \begin{cases} \frac{1}{2}, & t \in [1,3] \\ 0, & \text{otherwise} \end{cases} \\
f_{T|C}(t) &= \begin{cases} \frac{1}{3}, & t \in [0,3] \\ 0, & \text{otherwise} \end{cases} \\
f_T(t) &= f_{T|A}(t)P(A) + f_{T|B}(t)P(B) + f_{T|C}(t)P(C) \\
&= \begin{cases} \frac{13}{27}, & t \in [0,1] \\ \frac{7}{27}, & t \in [1,3] \\ 0, & \text{otherwise} \end{cases}
\end{aligned}$$

(d) Let x denote a location on the circle, where $x \in [0,6)$. Let $f(x)$ be a random indicator function of a point x where

$$f(x) = \begin{cases} 1, & x \text{ covered} \\ 0, & \text{otherwise} \end{cases}$$

Of course, the value of $f(x)$ will depend on the random locations of the units. The total random amount of the city that is covered at a random time is given by $\int_0^6 f(x)dx$. The average (i.e. expected) amount of the city that is covered at a random time is $E\left[\int_0^6 f(x)dx\right]$. Recall that the expected value of a sum of random variables is always the sum of the expected values, regardless of whether the random variables are independent or not (linearity of expectation). Since the integration inside the expectation is essentially the same as summation (we're just summing over tiny intervals), we have that

$$\begin{aligned} E\left[\int_0^6 f(x)dx\right] &= \int_0^6 E[f(x)]dx \\ &= \int_0^6 P(x \text{ covered})dx \end{aligned}$$

where the last equality follows from the fact that the expected value of an indicator function is equal to the probability that the indicator equals 1. Now, it is easy to see that $P(x \text{ covered})$ may not be the same for all values of $x \in [0, 6)$. for instance, if x equals the home location of one of the units, then the probability of coverage is likely to be higher. However, for certain intervals, it turns out that $P(x \text{ covered})$ is constant. In particular, we can break our analysis into two cases.

First, let us derive γ_{ij}, the fractions of i dispatches whose destination is district j, $i, j \in \{1, 2, 3\}$. Note that the γ_{ij} are not the same as the f_{ij} defined in the text, since the latter is given to be the fraction of *all* dispatches (not just i dispatches), which take unit i to district j. We will derive the γ_{ij} from the f_{ij} via renormalization. Recall that each f_{ij} is the sum of a term corresponding to the fraction of dispatches of i to j that incur no queueing delay and those that do incur a delay in queue. For any call delayed in queue (arrives when all of the units are busy), because of the memorylessness of the exponential distribution, the call is equally likely to be answered by any one of the three units.

$$\begin{aligned}
f_{11} &= P(\text{call arr to 1})\,(P(\text{1 free}) + P(\text{all busy, and 1 becomes free first})) \\
&= \frac{1}{3}\left(1-\rho_1+\frac{1}{3}(1-P_0-P_1-P_2)\right) \\
&= \frac{1}{3}\left(\frac{1}{3}+\frac{1}{3}\cdot\frac{4}{9}\right) \;=\; \frac{13}{81} \\
f_{ii} &= \frac{13}{81}, \forall i \in \{1,2,3\},\ \text{by symmetry} \\
f_{12} &= P(\text{call arr to 2})\left(P_{110}+\frac{1}{2}P_{010}+\frac{1}{3}(1-P_0-P_1-P_2)\right) \\
&= \frac{1}{3}\left(\frac{2}{27}+\frac{1}{2}\cdot\frac{2}{27}+\frac{1}{3}\cdot\frac{4}{9}\right) \\
&= \frac{7}{81} \\
f_{ij} &= \frac{7}{81},\ \forall i \neq j,\ \text{by symmetry}
\end{aligned}$$

So, to derive the γ_{ij}, we must renormalize so that the sample space includes only dispatches of unit i. Again, we can exploit the symmetry of the problem to conclude that

$$\begin{aligned}
\gamma_{ii} &= \frac{f_{11}}{f_{11}+f_{12}+f_{13}} = \frac{13}{27},\ \forall i \in \{1,2,3\} \\
\gamma_{ij} &= \frac{f_{12}}{f_{11}+f_{12}+f_{13}} = \frac{7}{27},\ \forall i \neq j
\end{aligned}$$

Now, we are ready to derive $P(x \text{ covered})$. By the symmetry of the problem, we derive this result by assuming, without loss of generality, that x is located within district 1. Now, we need to consider the following two cases. Let E_i denote the event that x is covered by unit i.

- **Case 1:** x is within $\frac{1}{2}$ mile of the home location of unit 1.

  To explain the entries in the following table, note that a unit located within district 1 covers

x if it is in the $\frac{1}{2}$ mile to the right or to the left of x. Thus, the unit must be in one of two subintervals whose lengths add to 1 mile. Given that a unit is in district 1, its exact location within the district is uniformly distributed. Therefore, given that a unit is in district 1, the probability that it covers x is simply $\frac{1}{2}$ (since district 1 has total length 2 miles). Let Q denote the state in which the number n of emergency calls in the system is at least 4.

States	$\Pr(E_1)$	$\Pr(E_2)$	$\Pr(E_3)$	$\Pr(x \text{ covered})$
000, 010, 100, 110	1	who cares	who cares	1
001	$\frac{\gamma_{11}}{2}$	0	0	$\frac{\gamma_{11}}{2}$
011	$\frac{\gamma_{11}}{2}$	$\frac{\gamma_{21}}{2}$	0	$1-\left(1-\frac{\gamma_{11}}{2}\right)\left(1-\frac{\gamma_{21}}{2}\right)$
101	$\frac{\gamma_{11}}{2}$	0	$\frac{\gamma_{31}}{2}$	$1-\left(1-\frac{\gamma_{11}}{2}\right)\left(1-\frac{\gamma_{31}}{2}\right)$
111, Q	$\frac{\gamma_{11}}{2}$	$\frac{\gamma_{21}}{2}$	$\frac{\gamma_{31}}{2}$	$1-\left(1-\frac{\gamma_{11}}{2}\right)\left(1-\frac{\gamma_{21}}{2}\right)\left(1-\frac{\gamma_{31}}{2}\right)$

The probability that x is covered, given Case 1, is

$$\begin{aligned} P(x \text{ covered} \mid \text{Case 1}) &= (1-\rho_1) + P_{001}\frac{\gamma_{11}}{2} \\ &\quad + P_{011}\left[1-\left(1-\frac{\gamma_{11}}{2}\right)\left(1-\frac{\gamma_{21}}{2}\right)\right] \\ &\quad + P_{101}\left[1-\left(1-\frac{\gamma_{11}}{2}\right)\left(1-\frac{\gamma_{31}}{2}\right)\right] \\ &\quad + (P_{111}+P_Q)\left[1-\left(1-\frac{\gamma_{11}}{2}\right)\left(1-\frac{\gamma_{21}}{2}\right)\left(1-\frac{\gamma_{31}}{2}\right)\right] \\ &= 0.5902 \end{aligned}$$

- **Case 2:** x is in district 1 but further than $\frac{1}{2}$ mile from the home location of unit 1.

Without loss of generality, we can assume that x is located no more than $\frac{1}{2}$ mile from the district 1-2 boundary (otherwise, the analysis is the same but involves districts 1 and 3, rather tan districts 1 and 2). Suppose that x is located in district 1, D units from the district 1-2 boundary. $D \sim \mathrm{U}\left(0, \frac{1}{2}\right)$.

Suppose we are given that $D = d$. A unit in district 1 covers x iff it lies in the d units between x and the district 1-2 boundary or lies in the $\frac{1}{2}$ mile on the other side of x. Thus, given that a unit is in district 1, and given d, the probability that it covers x is $\frac{\frac{1}{2}+d}{2}$, since district 1 has total length 2 miles. In contrast to Case 1, we now also have that a unit in district 2 may cover x, even though x is in district 1. In particular, a unit in district 2 covers x iff it is in the $\frac{1}{2} - d$ mile of district 2 that borders on the district 1-2 boundary. So, given that a unit is in district 2, and given d, the probability that it covers x is $\frac{\frac{1}{2}-d}{2}$.

However, the probabilities that we gave above are conditional probabilities, where we've conditioned on the distance d of x from the district 1-2 boundary. The corresponding unconditional probabilities are as follows. Given that a unit is in district 1, the probability that it covers x is $\int_0^{\frac{1}{2}} 2 \cdot \frac{\frac{1}{2}+u}{2} du = \frac{3}{8}$. Similarly, given that a unit is in district 2, the probability that it covers x is $\int_0^{\frac{1}{2}} 2 \cdot \frac{\frac{1}{2}-u}{2} du = \frac{1}{8}$.

States	$\Pr(E_1)$	$\Pr(E_2)$	$\Pr(E_3)$
000	0	0	0
001	$\frac{3}{8}\gamma_{11} + \frac{1}{8}\gamma_{12}$	0	0
010	0	$\frac{3}{8}\gamma_{21} + \frac{1}{8}\gamma_{22}$	0
100	0	0	$\frac{3}{8}\gamma_{31} + \frac{1}{8}\gamma_{32}$
011	$\frac{3}{8}\gamma_{11} + \frac{1}{8}\gamma_{12}$	$\frac{3}{8}\gamma_{21} + \frac{1}{8}\gamma_{22}$	0
101	$\frac{3}{8}\gamma_{11} + \frac{1}{8}\gamma_{12}$	0	$\frac{3}{8}\gamma_{31} + \frac{1}{8}\gamma_{32}$
110	0	$\frac{3}{8}\gamma_{21} + \frac{1}{8}\gamma_{22}$	$\frac{3}{8}\gamma_{31} + \frac{1}{8}\gamma_{32}$
111, Q	$\frac{3}{8}\gamma_{11} + \frac{1}{8}\gamma_{12}$	$\frac{3}{8}\gamma_{21} + \frac{1}{8}\gamma_{22}$	$\frac{3}{8}\gamma_{31} + \frac{1}{8}\gamma_{32}$

The probability that x is covered, given Case 2, is

$$\begin{aligned}
P(x \text{ covered} \mid \text{Case 1}) &= P_{001}\left(\frac{3}{8}\gamma_{11} + \frac{1}{8}\gamma_{12}\right) \\
&\quad + P_{010}\left(\frac{3}{8}\gamma_{21} + \frac{1}{8}\gamma_{22}\right) + P_{100}\left(\frac{3}{8}\gamma_{31} + \frac{1}{8}\gamma_{32}\right) \\
&\quad + P_{011}\left[1 - \left(1 - \frac{3}{8}\gamma_{11} - \frac{1}{8}\gamma_{12}\right)\left(1 - \frac{3}{8}\gamma_{21} - \frac{1}{8}\gamma_{22}\right)\right] \\
&\quad + P_{110}\left[1 - \left(1 - \frac{3}{8}\gamma_{21} - \frac{1}{8}\gamma_{22}\right)\left(1 - \frac{3}{8}\gamma_{31} - \frac{1}{8}\gamma_{32}\right)\right] \\
&\quad + (P_{111} + P_Q)\left[1 - \left(1 - \frac{3}{8}\gamma_{11} - \frac{1}{8}\gamma_{12}\right)\left(1 - \frac{3}{8}\gamma_{21} - \frac{1}{8}\gamma_{22}\right)\right. \\
&\quad \left.\left(1 - \frac{3}{8}\gamma_{31} - \frac{1}{8}\gamma_{32}\right)\right] \\
&= 0.2697
\end{aligned}$$

Now, we are ready to compute the average length of the city that is covered at a random time. Note that there are 3 miles of the city that fit Case 1, namely the three 1-mile strips centered at each home location. At any point on these 3 miles, the probability of coverage is 0.5902. The remaining 3 miles of the city fit Case 2. The probability of coverage for any point along these remaining 3 miles is 0.2697. Let "$\int_{\text{Case k}}$" denote that we integrate over points along the city that fit Case k, for $k \in \{1, 2\}$. Therefore, the average length of the city that is covered at a random time is given by

$$\begin{aligned} E\left[\int_0^6 f(x)dx\right] &= \int_0^6 P(x \text{ covered})dx \\ &= 0.5902 \int_{\text{Case 1}} dx + 0.2697 \int_{\text{Case 2}} dx \\ &= 3\,(0.5902 + 0.2697) \;=\; 2.5797 \end{aligned}$$