1. (Aghassi 2002)

Let $G(a) = E[Z^2]$, where $Z = \max(X_1, \ldots, X_7)$, $X_i \sim \mathrm{U}(0,a)$ i.i.d. $i \in \{1, \ldots, 7\}$. Let us calculate $G(a+\epsilon)$ for the following cases.

Case	Probability of Case	$G(a+\epsilon)$ given case
$X_i \in [0,a]$, $\forall i \in \{1,\ldots,7\}$	$\left(\frac{a}{a+\epsilon}\right)^7$	$G(a)$
Exactly 1 of the $X_i \in (a, a+\epsilon]$ The other 6 in $[0,a]$	$7\frac{\epsilon a^6}{(a+\epsilon)^7}$	$\int_a^{a+\epsilon} \frac{1}{\epsilon} x^2 dx$ $= \frac{1}{3\epsilon}\left(3a^2\epsilon + 3a\epsilon^2 + \epsilon^3\right)$ $= a^2 + a\epsilon + o(\epsilon)$
More than 1 of the $X_i \in (a, a+\epsilon]$ The rest in $[0,a]$	$o(\epsilon)$	"Who cares"

where $h(\epsilon)$ is said to be $o(\epsilon)$ iff $\lim_{\epsilon \to 0} \frac{h(\epsilon)}{\epsilon} = 0$. That is, $h(\epsilon)$ goes to zero faster than does ϵ as ϵ approaches zero. Let us examine the probabilities shown in this table. Since the X_i are i.i.d. $U(0, a+\epsilon)$, the event that $X_i \in [a, a+\epsilon]$ can be viewed as a Bernoulli trial with probability of success $\frac{\epsilon}{a+\epsilon}$. So, the probability that exactly k of them are in this interval is given by the probability that a Binom$\left(7, \frac{\epsilon}{a+\epsilon}\right)$ RV takes on value k. That is, the probability is given by $\begin{pmatrix} 7 \\ k \end{pmatrix} \left(\frac{\epsilon}{a+\epsilon}\right)^k \left(\frac{a}{a+\epsilon}\right)^{7-k}$. The reason why we can say "who cares" for the cases when more than 1 of the X_i falls in $[a, a+\epsilon]$ is that $G(a+\epsilon) \leq (a+\epsilon)^2$. And, $o(\epsilon)(a+\epsilon)^2 = o(\epsilon)$. So, by conditioning on each of the given cases, we can say that

$$G(a+\epsilon) \quad = \quad \left(\frac{a}{a+\epsilon}\right)^7 G(a) + 7\frac{\epsilon a^6}{(a+\epsilon)^7}\left(a^2 + a\epsilon\right) + o(\epsilon)$$

Now let us use a Taylor expansion to rewrite $\left(\frac{a}{a+\epsilon}\right)^7$ and $7\frac{\epsilon a^6}{(a+\epsilon)^7}$.

$$\begin{aligned}
\left(\frac{a}{a+\epsilon}\right)^7 &= \left(\frac{1}{1+\frac{\epsilon}{a}}\right)^7 \\
&= \left(1 - \frac{\epsilon}{a} + \left(\frac{\epsilon}{a}\right)^2 - \left(\frac{\epsilon}{a}\right)^3 + \cdots\right)^7 \\
&= 1 - 7\frac{\epsilon}{a} + o(\epsilon) \\
7\frac{\epsilon a^6}{(a+\epsilon)^7} &= 7\frac{\epsilon}{a}\left(\frac{a}{a+\epsilon}\right)^7 \\
&= 7\frac{\epsilon}{a} + o(\epsilon)
\end{aligned}$$

Plugging back into the equation for $G(a+\epsilon)$, we obtain

$$\begin{aligned}
G(a+\epsilon) &= \left(1 - 7\frac{\epsilon}{a}\right) G(a) + 7\frac{\epsilon}{a}\left(a^2 + a\epsilon\right) + o(\epsilon) \\
&= \left(1 - 7\frac{\epsilon}{a}\right) G(a) + 7a\epsilon + o(\epsilon) \\
\frac{G(a+\epsilon) - G(a)}{\epsilon} &= -\frac{7}{a}G(a) + 7a + \frac{o(\epsilon)}{\epsilon} \\
\lim_{\epsilon \to 0} \frac{G(a+\epsilon) - G(a)}{\epsilon} = G'(a) &= -\frac{7}{a}G(a) + 7a
\end{aligned}$$

To solve this differential equation, let us start by multiplying through by a^7.

$$\begin{aligned}
a^7 G'(a) + 7a^6 G(a) &= 7a^8 \\
\left[a^7 G(a)\right]' &= 7a^8
\end{aligned}$$

Integrating both sides yields $a^7 G(a) = \frac{7}{9}a^9 + K$, where K is the constant of integration. It is obvious that $G(0) = 0$, since $Z = 0$, whenever $X_i = \mathrm{U}(0,0)$, $i \in \{1, \ldots, 7\}$ i.i.d. We can use this boundary condition to solve for K. Namely, $0 = 0 + K$. Therefore $K = 0$. So, finally

$$G(a) = \frac{7}{9}a^2$$

2 (Kang, 2001)

For this problem, we employ the notation used in class, which is a little different from the notation in the textbook.

Let $G(a) \equiv E[D^p] \equiv E[|X_1 - X_2|^p]$. Let us consider $G(a+\varepsilon)$ that is $E[D^p]$ when the highway segment under consideration is extended by ε where ε is very small. Suppose $a < X_1 \leq a + \varepsilon$ and

$0 \le X_2 \le a$. Since X_1 and X_2 are independent, $G(a+\varepsilon)$ for this case is computed as follows:

$$G(a+\varepsilon) = E[(X_1 - X_2)^p] = \int_a^{a+\varepsilon} \int_0^a (x_1 - x_2)^p f_{X_2}(x_2) f_{X_1}(x_1)\, dx_2\, dx_1\,,$$

where $f_{X_1}(x_1)$ and $f_{X_2}(x_2)$ are the probability density functions of X_1 and X_2, respectively. Because X_1 and X_2 are uniformly distributed over $(a, a+\varepsilon]$ and $[0,a]$ respectively, $f_{X_1}(x_1) = \frac{1}{\varepsilon}$ and $f_{X_2}(x_2) = \frac{1}{a}$. Thus,

$$\begin{aligned}
G(a+\varepsilon) &= \frac{1}{a\varepsilon} \int_a^{a+\varepsilon} \int_0^a (x_1 - x_2)^p\, dx_2\, dx_1 \\
&= \frac{1}{a\varepsilon} \int_a^{a+\varepsilon} \left[\frac{-1}{p+1} (x_1 - x_2)^{p+1} \right]_0^a dx_1 \\
&= \frac{1}{a\varepsilon} \cdot \frac{1}{p+1} \int_a^{a+\varepsilon} \left(x_1^{p+1} - (x_1 - a)^{p+1} \right) dx_1 \\
&= \frac{1}{a\varepsilon} \cdot \frac{1}{p+1} \left[\frac{1}{p+2} x_1^{p+2} - \frac{1}{p+2} (x_1 - a)^{p+2} \right]_a^{a+\varepsilon} \\
&= \frac{1}{a\varepsilon} \cdot \frac{1}{(p+1)(p+2)} \left((a+\varepsilon)^{p+2} - \varepsilon^{p+2} - a^{p+2} \right) \\
&= \frac{1}{a\varepsilon} \cdot \frac{1}{(p+1)(p+2)} \left((p+2) a^{p+1} \varepsilon + o(\varepsilon) \right),
\end{aligned}$$

where $o(\varepsilon)$ represents higher order terms of ε satisfying $\lim_{\varepsilon \to 0} \frac{o(\varepsilon)}{\varepsilon} = 0$ ("pathetic terms"). Clearly, $G(a+\varepsilon) \approx \frac{a^p}{(p+1)}$ as $\varepsilon \to 0$.

When $0 \le X_1 \le a$ and $a < X_2 \le a+\varepsilon$, we also have $G(a+\varepsilon) \approx \frac{a^p}{(p+1)}$ as $\varepsilon \to 0$ by symmetry. If $0 \le X_1 \le a$ and $0 \le X_2 \le a$, then $G(a+\varepsilon) = G(a)$. Finally, we do not have to compute $G(a+\varepsilon)$ for the case where $a < X_1 \le a+\varepsilon$ and $a < X_2 \le a+\varepsilon$ because the associated probability is negligible. The following table summarizes $G(a+\varepsilon)$'s.

Case	Probability of a case	$G(a+\varepsilon)$ given a case
$0 \le X_1 \le a,\ 0 \le X_2 \le a$	$\frac{a}{a+\varepsilon} \cdot \frac{a}{a+\varepsilon} = (\frac{a}{a+\varepsilon})^2$	$G(a)$
$a < X_1 \le a+\varepsilon,\ 0 \le X_2 \le a$	$\frac{\varepsilon}{a+\varepsilon} \cdot \frac{a}{a+\varepsilon} = \frac{\varepsilon a}{(a+\varepsilon)^2}$	$\frac{a^p}{(p+1)}$
$0 \le X_1 \le a,\ a < X_2 \le a+\varepsilon$	$\frac{a}{a+\varepsilon} \cdot \frac{\varepsilon}{a+\varepsilon} = \frac{\varepsilon a}{(a+\varepsilon)^2}$	$\frac{a^p}{(p+1)}$
$a < X_1 \le a+\varepsilon,\ a < X_2 \le a+\varepsilon$	$\frac{\varepsilon}{a+\varepsilon} \cdot \frac{\varepsilon}{a+\varepsilon} = (\frac{\varepsilon}{a+\varepsilon})^2$	We do not care.

Using the total expectation theorem, we obtain

$$\begin{aligned} G(a+\varepsilon) &= G(a)\left(\frac{a}{a+\varepsilon}\right)^2 + \frac{a^p}{(p+1)}\frac{\varepsilon a}{(a+\varepsilon)^2} + \frac{a^p}{(p+1)}\frac{\varepsilon a}{(a+\varepsilon)^2} + o(\varepsilon^2) \\ &= G(a)\left(\frac{a}{a+\varepsilon}\right)^2 + \frac{2a^p}{(p+1)}\frac{\varepsilon a}{(a+\varepsilon)^2} + o(\varepsilon^2) \\ &\approx G(a)\left(\frac{a}{a+\varepsilon}\right)^2 + \frac{2a^p}{(p+1)}\frac{\varepsilon a}{(a+\varepsilon)^2}\,. \end{aligned}$$

From the formula of the sum of an infinite geometric series, we know

$$\frac{a}{a+\varepsilon} = \frac{1}{1+\frac{\varepsilon}{a}} = 1 - \frac{\varepsilon}{a} + \left(\frac{\varepsilon}{a}\right)^2 - \left(\frac{\varepsilon}{a}\right)^3 + \cdots\,.$$

Ignoring higher order terms of ε, we get

$$\frac{a}{a+\varepsilon} \approx 1 - \frac{\varepsilon}{a}\,.$$

This gives the following approximations:

$$\left(\frac{a}{a+\varepsilon}\right)^2 \approx \left(1-\frac{\varepsilon}{a}\right)^2 = 1 - \frac{2\varepsilon}{a} + \frac{\varepsilon^2}{a^2} \approx 1 - \frac{2\varepsilon}{a}\,,$$

$$\frac{\varepsilon a}{(a+\varepsilon)^2} = \frac{\varepsilon}{a}\left(\frac{a}{a+\varepsilon}\right)^2 \approx \frac{\varepsilon}{a}\left(1-\frac{2\varepsilon}{a}\right) = \frac{\varepsilon}{a} - \frac{2\varepsilon^2}{a^2} \approx \frac{\varepsilon}{a}\,.$$

Therefore, we can rewrite $G(a+\varepsilon)$ as

$$G(a+\varepsilon) \approx G(a)\left(1-\frac{2\varepsilon}{a}\right) + \frac{2a^p}{(p+1)}\cdot\frac{\varepsilon}{a} = G(a)\left(1-\frac{2\varepsilon}{a}\right) + \frac{2a^{p-1}\varepsilon}{(p+1)}\,.$$

Rearranging terms, we have

$$\frac{G(a+\varepsilon)-G(a)}{\varepsilon} = -\frac{2G(a)}{a} + \frac{2a^{p-1}}{(p+1)}\,.$$

If $\varepsilon \to 0$, we have the following differential equation:

$$G'(a) = -\frac{2G(a)}{a} + \frac{2a^{p-1}}{(p+1)}\,.$$

"Judicious" guesses (or consultation with books on differential equations) lead us to the following solution:

$$G(a) \equiv E[D^p] = \frac{2a^p}{(p+1)(p+2)}\,.$$

We can skip the derivation of the differential equation by directly using Equation (3.64) in the

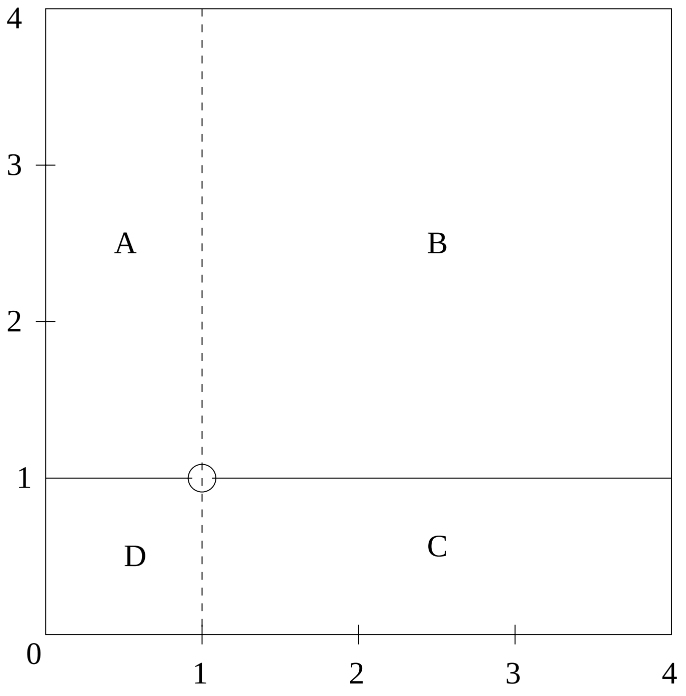

Figure 1: Barrier Configuration

textbook. Once we obtain $G(a+\varepsilon)$, we can plug it in (3.64), which gives the same differential equation as above.

3 (Aghassi 2002)

The situation is illustrated in Figure 1. To solve the problem, let us divide the region into four subregions by adding a dashed vertical line (not representing a barrier) at $x = 1$.

(i) First, let us note that the barrier does not cause extra y travel distance. It can, however, cause extra x travel distance. For example, extra x travel distance will be required for travel between points in regions A and D and between points in regions B and C. The maximum extra x travel distance will be incurred when, without the barrier, the x travel distance would be 0, but with the barrier, the x travel distance is two times the length of the longer side of the barrier. Since the longer side of the barrier has length three, the maximum possible extra travel distance will be 6. This situation arises whenever the emergency and response vehicle both have x-coordinate 4, but one is in region B, while the other is in region C.

(ii) Let D_e denote the additional travel distance due to the barrier. By reasoning similar to that used in part (i), note that if the emergency is in region A and the response unit is in region

D, or vice versa, the maximum possible extra travel distance will be 2. So, in order for $D_e > 5$, we must have the emergency in region B and the response unit in region C, or vice versa. Let BC denotes the event that one of the emergency and the response unit is in region B and the other is in region C. Then,

$$\begin{aligned} P(D_e > 5) &= P(D_e > 5 \mid BC)P(BC) \\ P(BC) &= P(\text{emergency in B, unit in C}) + P(\text{emergency in C, unit in B}) \\ &= 2P(\text{emergency in B, unit in C}) \\ & \text{ since the locations of emergency and unit are identically distrib} \\ &= 2P(\text{emergency in B})P(\text{unit in C}) \\ & \text{ since the locations of emergency and unit are indep} \\ &= 2 \cdot \frac{9}{16} \cdot \frac{3}{16} \end{aligned}$$

Let X_u and X_i denote the x-coordinates of the response unit and emergency, respectively. Given event BC, the x travel distance without the barrier is $\mid X_u - X_i \mid$. With the barrier, it is $X_u - 1 + X_i - 1$. So, the extra travel distance is

$$\begin{aligned} (X_u - 1) + (X_i - 1) - \mid X_u - X_i \mid &= (X_u - 1) + (X_i - 1) - \mid (X_u - 1) - (X_i - 1) \mid \\ &= 2 \min(X_u - 1, X_i - 1) \end{aligned}$$

Accordingly, given event BC, $D_e > 5$ whenever $2\min(X_u - 1, X_i - 1) > 5 \iff X_u, X_i > 3.5$.

$$\begin{aligned} P(X_u > 3.5, X_i > 3.5 \mid BC) &= P(X_u > 3.5 \mid BC)P(X_i > 3.5 \mid BC), \text{ since } X_u,\ X_i \text{ indep} \\ &= \left(\frac{0.5}{3}\right)^2 \end{aligned}$$

The last equality follows since X_u and X_i are identically distributed, and since, given BC, they are both $\mathrm{U}(1, 4)$. Therefore,

$$P(D_e > 5) = \frac{1}{36} \cdot \left(2 \cdot \frac{9}{16} \cdot \frac{3}{16}\right) = \frac{3}{512}$$

♣ <u>ASIDE:</u> *It is tempting, but incorrect, to argue that* $D_e > 5$ *whenever* $X_u - 1 + X_i - 1 > 5$. *That is, whenever* $X_u + X_i > 7$. *In fact, nearly half the class made this mistake.* $X_u - 1 + X_i - 1$ *gives the total* x*-distance traveled with the barrier, which is not always equal to the* <u>*extra*</u> x*-distance traveled. To see why, consider* $(X_u, X_i) = (3.1, 4)$, *which satisfies* $X_u + X_i > 7$. *Without the barrier, the* x*-distance would be 0.9. With the barrier, the* x*-distance is* $(3.1 - 1) + (4 - 1) = 5.1$. *So, the extra travel distance is only 4.2. So, while this condition captures some cases when* $D_e > 5$, *it also captures several cases where* $D_e < 5$.

(iii) We already noted that there is no additional travel distance due to the barrier unless either of the following two cases hold

- one of the response unit and emergency is in region A and the other is in region D. Let AD denote this event. By reasoning similar to that used in part (ii), we have that $P(AD) = 2P(\text{emergency in A})P(\text{unit in D}) = 2 \cdot \frac{3}{16} \cdot \frac{1}{16} \;=\; \frac{3}{128}$.
- one of the response unit and emergency is in region B and the other is in region C. Let BC denote this event. We already calculated $P(BC) = \frac{27}{128}$.

Using conditional expectations, we can say that

$$\begin{aligned} E[D_e] \;&=\; E[D_e \mid BC]P(BC) + E[D_e \mid AD]P(AD) \\ &\quad + E[D_e \mid \text{neither } BC \text{ nor } AD]P(\text{neither } BC \text{ nor } AD) \end{aligned}$$

Let X_u and Y_u denote the x- and y-coordinates, repsectively, of the response unit, and X_i and Y_i denote the x- and y-coordinates, respectively, of the emergency.

Case 1) Event BC

To calculate expected travel distances in the case BC, we can assume without loss of generality that the response unit is in region B and the emergency is in region C. The justification is that the travel distance from a point in region B to a point in region C is equivalent to the distance from the same point in region C to the same point in region B. Given this case and these additional specifics, $X_u, X_i \sim \mathrm{U}(1,4)$ i.i.d., $Y_u \sim \mathrm{U}(1,4)$ and $Y_i \sim \mathrm{U}(0,1)$. Without the barrier, the expected travel distance is

$$\begin{aligned} E[\mid X_u - X_i \mid] + E[Y_u - Y_i] \;&=\; \frac{1}{3}3 + E[Y_u] - E[Y_i] \\ &=\; 1 + 2.5 - 0.5 \;=\; 3 \end{aligned}$$

With the barrier, the expected travel distance is

$$E[X_u - 1] + E[X_i - 1] + E[Y_u - Y_i] \;=\; (2.5-1) + (2.5-1) + 2 \;=\; 5$$

Therefore, $E[D_e \mid BC] = 5 - 3 = 2$.

Case 2) Event AD

As in the preceding case, we can assume without loss of generality that the response unit is in region A and the emergency is in region D. Then, $X_u, X_i \sim \mathrm{U}(0,1)$ i.i.d., $Y_u \sim \mathrm{U}(1,4)$ and $Y_i \sim \mathrm{U}(0,1)$. Without the barrier, the expected travel distance is

$$E[\mid X_u - X_i \mid] + E[Y_u - Y_i] \;=\; \frac{1}{3} + 2.5 - 0.5 \;=\; 2\frac{1}{3}$$

With the barrier, the expected travel distance is

$$E[1-X_u]+E[1-X_i]+E[Y_u-Y_i] \;=\; (1-0.5)+(1-0.5)+2 \;=\; 3$$

Therefore, $E[D_e \mid AD] = 3 - 2\frac{1}{3} = \frac{2}{3}$.

Plugging back into our equation for $E[D_e]$, and noting that $E[D_e \mid \text{neither } BC \text{ nor } AD] = 0$, we obtain

$$\begin{aligned} E[D_e] &= E[D_e \mid BC]P(BC) + E[D_e \mid AD]P(AD) \\ &= 2 \cdot \frac{27}{128} + \frac{2}{3} \cdot \frac{3}{128} \\ &= \frac{7}{16} \end{aligned}$$

4 (EC 1997)

Following the notation given in the text, (X_1, Y_1) and (X_2, Y_2) denote the locations of the response unit and incident respectively. $S(S^{'})$ denote the set of points within(outside) the central square; and, $A = \{(X_1, Y_1) \in S\}$ and $B = \{(X_2, Y_2 \in S\}$.

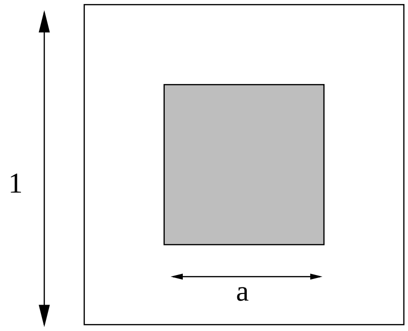

Figure 2: Zero-demand zone and unit square response area

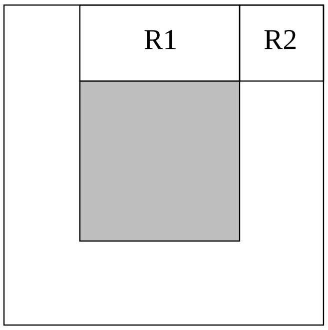

Figure 3: Zero-demand zone and unit square response area

a. Considering a unit square on which incidents and reponse unit are uniformly, independently distributed aver the *entire* system, the expected travel distance:

$$\begin{aligned}
E[D] &= E[D|A\cap B]P[A\cap B]+E[D|A\cap B']P[A\cap B'] \\
&\quad +E[D|A'\cap B]P[A'\cap B]+E[D|A'\cap B']P[A'\cap B'] \\
&= E[D|A\cap B]P[A\cap B]+2E[D|A\cap B']P[A\cap B'] \\
&\quad +E[D|A'\cap B']P[A'\cap B']
\end{aligned}$$

We know that $E[D]=\frac{2}{3}$ from class and $P(A)=P(B)=a^2$. And since A and B are independent,

$$\begin{aligned}
E[D] &= E[D|A\cap B]P(A)P(B)+2E[D|A\cap B']P(A)P(B') \\
&\quad +E[D|A'\cap B']P(A')P(B') \\
&= \frac{2}{3}a(a^2)^2+2E[D|A\cap B']a^2(1-a^2)+E[D|A'\cap B'](a-a^2)^2
\end{aligned}$$

b. (i) The set B' consists of (X_2,Y_2) outside the shaded square which can be divided into two classes of identically-sized shapes: thos of type R_1 (bordering the square) and those of type R_2 (at corners of the square). Hence,

$$E[D|A\cap B']=4E[D|A\cap R_1]\cdot P(R_1)+4E[D|A\cap R_2]\cdot P(R_2)$$

(ii)

$$\begin{aligned}
&P\{(X_2,Y_2)\in R_1|(X_2,Y_2)\in(R_1\cup R_2)\} \\
&= \frac{P\{(X_2,Y_2)\in R_1\}}{P\{(X_2,Y_2)\in R_1\cup R_2\}}=\frac{a(1-a)\frac{1}{2}}{\frac{1}{2}a(1-a)+[\frac{1}{2}(1-a)]^2} \\
&= \frac{a}{a+\frac{1}{2}(1-a)}=\frac{2a}{1+a}
\end{aligned}$$

(iii)

$$\begin{aligned}
E[D|A\cap R_1] &= \left\{\frac{1}{2}\left[\frac{1}{2}(1-a)\right]+\frac{1}{2}a\right\}+\frac{1}{3}a \\
&= \left\{\frac{1}{4}(1-a)+\frac{1}{2}a\right\}+\frac{1}{3}a \\
&= \frac{1}{4}(1-a)+\frac{1}{3}a=\frac{1}{4}+\frac{7}{12}a \\
E[D|A\cap R_2] &= 2\left\{\frac{1}{2}\left[\frac{1}{2}(1-a)\right]+\frac{1}{2}a\right\}=\frac{1}{2}(1+a)
\end{aligned}$$

c.

$$\begin{aligned}
\bar{W}(a) &= E[D|A^{'} \cap B^{'}] \\
&= \frac{E(D) - \frac{2}{3}a(a^2)^2 - 2E[D|A \cap B^{'}]a^2(1-a^2)}{(1-a^2)^2} \\
E[D|A \cap B^{'}] &= E[D|A \cap R_1]P(R_1) + E[D|A \cap R_2]P(R_2) \\
&= \left(\frac{1}{4} + \frac{7}{12}a\right)\left(\frac{2a}{a+1}\right) + \left(\frac{1}{2} + \frac{1}{2}a\right)\left(1 - \frac{2a}{a+1}\right) \\
&= \left(\frac{1}{2}\right)\left[\left(1 + \frac{7}{3}a\right)\left(\frac{a}{a+1}\right) + (a+1)\left(\frac{1-a}{a+1}\right)\right] \\
&= \frac{1}{2(a+1)}\left[a\left(1 + \frac{7}{3}a\right) + (a+1)(1-a)\right] \\
&= \frac{1}{2(a+1)}\left(\frac{4}{3}a^2 + a + 1\right) \\
\Rightarrow \bar{W}(a) &= \frac{\frac{2}{3}(1-a^5) - a^2(1-a)(\frac{4}{3}a^2 + a + 1)}{(1-a^2)^2} \\
&= \frac{2 + 2a - a^2 - a^3 - 2a^4}{3(1+a)(1-a^2)}
\end{aligned}$$

Clearly, $\bar{W}(0) = \frac{2}{3}$. Since $\bar{W}(1)$ is undefined, by l'Hospital's Rule:

$$\begin{aligned}
\lim_{a \to 1} \bar{W}(a) &= \left.\frac{\frac{d}{da}(2 + 2a - a^2 - a^3 - 2a^4)}{\frac{d}{da}3(1+a)(1-a^2)}\right|_{a=1} \\
&= \left.\frac{2 - 2a - 3a^2 - 8a^3}{3(1 - 2a - 3a^2)}\right|_{a=1} = \frac{11}{12}
\end{aligned}$$

$\bar{W}(1)$ is the average distance from any point in the unit square to its perimeter.

5 (Aghassi 2002)

(i) What is the probability that an eastbound plane will be in conflict with exactly 3 northbound planes?

In class, we showed that an eastbound plane is at any time in conflict with a northbound plane if the following condition holds. At the time the eastbound plane is at the junction, the northbound plane is within $5\sqrt{2}$ of it. So, an eastbound plane will have been in conflict at some point with each of exactly three different northbound planes if, at the time the eastbound plane is at the junction, there are exactly three northbound planes within $5\sqrt{2}$ of it. Recall that, in this model, the arrival process of northbound planes to the junction is taken to be Poisson with rate λ_N per minute. Furthermore, planes travel at 600 mph = 10 miles/min. So, when the eastbound plane is at the junction, there will be exactly 3 northbound planes within $5\sqrt{2}$ of it iff exactly 3 northbound planes will arrive at the junction in a $\frac{10\sqrt{2}}{10} = \sqrt{2}$ minute time horizon. That is

$$P(\text{conflict with exactly 3 N planes}) = \frac{e^{-\lambda_N\sqrt{2}}(\lambda_N\sqrt{2})^3}{3!}$$

(ii) Given such a triple conflict, what is the probaiblity that all three northbound planes are in conflict with one another?

From section 2.12.3 in the Larson and Odoni textbook, we know that the unordered arrival times in a Poisson process are independently, uniformly distributed over the fixed time interval of interest. So, in our case, given that there are 3 northbound arrivals in time interval $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$, the arrival times of these three planes are independently and uniformly distributed over this interval. Since distance is just speed multiplied by time, we can equivalently say that, given that there are 3 northbound planes located spatially in an interval $(-5\sqrt{2}, 5\sqrt{2})$, their locations within this interval are independently and uniformly distributed over the interval. Finally, to make the calculations easier, we can work with the interval $(0, 10\sqrt{2})$, since shifting the endpoints in this way will not affect the probability calculation.

Let X_1, X_2, and X_3 be the locations of the planes in this interval. We know that $X_i \sim \mathrm{U}(0, 10\sqrt{2})$ $i \in \{1, 2, 3\}$ i.i.d. All three of these planes will be in conflict with one another if every pair of planes chosen from the 3 are within 5 miles of each other. This occurs iff $\max(X_1, X_2, X_3) - \min(X_1, X_2, X_3) \leq 5$. For ease of notation, let X_{min} denote $\min(X_1, X_2, X_3)$, X_{max} denote $\max(X_1, X_2, X_3)$ and X_{mid} denote the remaining point.

There are at least two ways to solve the problem.

Method 1 Work with the sample space directly to obtain relevant derived distributions.

We want $P(X_{\text{max}} - X_{\text{min}} \leq 5)$. We can derive the CDF of X_{min} as follows. For $r \in [0, 10\sqrt{2}]$,

$$\begin{aligned}
P(X_{\text{min}} \leq r) &= 1 - P(\min(X_1, X_2, X_3) > r) \\
&= 1 - P(X_1 > r, X_2 > r, X_3 > r) \\
&= 1 - P(X_1 > r)P(X_2 > r)P(X_3 > r), \text{ by independence of } X_1, X_2, X_3 \\
&= 1 - \left(\frac{10\sqrt{2} - r}{10\sqrt{2}}\right)^3
\end{aligned}$$

For $r < 0$, $P(X_{\text{min}} \leq r) = 0$. And, for $r > 10\sqrt{2}$, $P(X_{\text{min}} \leq r) = 1$. Then

$$f_{X_{\text{min}}}(r) = \begin{cases} \frac{3}{10\sqrt{2}}\left(\frac{10\sqrt{2}-r}{10\sqrt{2}}\right)^2, & r \in [0, 10\sqrt{2}] \\ 0, & \text{otherwise} \end{cases}$$

Now, let us derive $P(X_{\max} - X_{\min} \leq 5)$ by conditioning on $X_{\min}$.

$$\begin{aligned} P(X_{\max} - X_{\min} \leq 5) &= \int_0^{10\sqrt{2}} P(X_{\max} - X_{\min} \leq 5 \mid X_{\min} = r) f_{X_{\min}}(r) dr \\ &= \int_0^{10\sqrt{2}} P(X_{\max} \leq 5 + r \mid X_{\min} = r) f_{X_{\min}}(r) dr \end{aligned}$$

We must derive $P(X_{\max} \leq x \mid X_{\min} = r)$. When $X_{\min} = r$, the remaining two points must be in $[r, 10\sqrt{2}]$. Since all X_i, $i \in \{1, 2, 3\}$, were uniformly distributed over the original interval $[0, 10\sqrt{2}]$, these remaining two points will be uniformly distributed over this new interval $[r, 10\sqrt{2}]$. Given that $X_{\min} = r$, $X_{\max} \leq x$ will never happen if $x < r$. If $x \geq r$, it will happen whenever the two non-minimum points are in $[r, x]$.

$$P(X_{\max} \leq x \mid X_{\min} = r) = \begin{cases} \left(\frac{x-r}{10\sqrt{2}-r}\right)^2, & x \in [r, 10\sqrt{2}] \\ 1, & x > 10\sqrt{2} \\ 0, & x < r \end{cases}$$

Plugging this formula back into the equation for $P(X_{\max} - X_{\min} \leq 5)$, we obtain

$$\begin{aligned} P(X_{\max} - X_{\min} \leq 5) &= \int_0^{10\sqrt{2}} P(X_{\max} \leq 5 + r \mid X_{\min} = r) f_{X_{\min}}(r) dr \\ &= \int_0^{10\sqrt{2}-5} \left(\frac{5}{10\sqrt{2} - r}\right)^2 \cdot \frac{3}{10\sqrt{2}} \left(\frac{10\sqrt{2} - r}{10\sqrt{2}}\right)^2 dr \\ &\quad + \int_{10\sqrt{2}-5}^{10\sqrt{2}} 1 \cdot \frac{3}{10\sqrt{2}} \left(\frac{10\sqrt{2} - r}{10\sqrt{2}}\right)^2 dr \\ &= 0.2866 \end{aligned}$$

Method 2 Use Crofton's Method, noting that a probability is itself the expected value of a relevant indicator RV.

Recall that Crofton's Method was presented in the text and in class as a method for computing mean values related to N points independently and uniformly distributed over some interval. We want to compute $P(X_{\max} - X_{\min} \leq 5)$. Let I be an indicator RV s.t.

$$I = \begin{cases} 1, & X_{\max} - X_{\min} \leq 5 \\ 0, & \text{otherwise} \end{cases}$$

And $P(X_{\max} - X_{\min} \leq 5) = E[I]$. Therefore, we can also use Crofton's Method to compute $P(X_{\max} - X_{\min} \leq 5)$. Let $X_i \sim \mathrm{U}(0, a)$ $i \in \{1, 2, 3\}$ i.i.d. And let $X_{\min} = \min(X_1, X_2, X_3)$, $X_{\max} = \max(X_1, X_2, X_3)$, and X_{mid} be the remaining point. Let $G(a) = P(X_{\max} - X_{\min} \leq 5)$.

We will consider only $a \geq 5$, since $G(a) = 1 \; \forall a \in [0,5]$.

Case	Probability of Case	$G(a+\epsilon)$ given case
$X_i \in [0,a]$, $\forall i \in \{1,2,3\}$	$\left(\frac{a}{a+\epsilon}\right)^3$	$G(a)$
Exactly 1 of the $X_i \in (a, a+\epsilon]$ The other 2 in $[0,a]$	$3\frac{\epsilon a^2}{(a+\epsilon)^3}$	$o(\epsilon)$ + Pr(two U$(0,a) \in [a-5,a]$) $= \left(\frac{5}{a}\right)^2 + o(\epsilon)$
More than 1 of the $X_i \in (a, a+\epsilon]$ The rest in $[0,a]$	$o(\epsilon)$	"Who cares"

Let us rigorously derive the result that $G(a+\epsilon)$ equals $\left(\frac{5}{a}\right)^2 + o(\epsilon)$ when exactly one of the X_i is in $(a, a+\epsilon]$. Without loss of generality, say $X_3 \in (a, a+\epsilon]$, $X_1, X_2 \in [0,a]$ (we could renumber the points if this were not the case). Then

$$\begin{aligned}
P(X_{\max} - X_{\min} \leq 5) &= \int_a^{a+\epsilon} P(X_3 - X_1 \leq 5, X_3 - X_2 \leq 5) f_{X_3}(u)\,du \\
&= \int_a^{a+\epsilon} P(X_3 - X_1 \leq 5) P(X_3 - X_2 \leq 5) f_{X_3}(u)\,du, \\
&\quad \text{by independence} \\
&= \frac{1}{\epsilon} \int_a^{a+\epsilon} P(X_1 \geq u-5) P(X_2 \geq u-5)\,du \\
&= \frac{1}{\epsilon} \int_a^{a+\epsilon} \left(\frac{a-u+5}{a}\right)^2 du \\
&= \frac{1}{3a^2\epsilon} \left[-(a-u+5)^3\right]_a^{a+\epsilon} \\
&= \frac{1}{3a^2\epsilon} \left(5^3 - (5-\epsilon)^3\right) \\
&= \frac{1}{3a^2\epsilon} \left(75\epsilon - 15\epsilon^2 + \epsilon^3\right) \\
&= \frac{25}{a^2} + o(\epsilon)
\end{aligned}$$

By the same approach as was used in problem 1, we can use Taylor expansions to obtain

$$\begin{aligned}
\left(\frac{a}{a+\epsilon}\right)^3 &= 1 - 3\frac{\epsilon}{a} + o(\epsilon) \\
3\frac{\epsilon a^2}{(a+\epsilon)^3} &= 3\frac{\epsilon}{a} + o(\epsilon)
\end{aligned}$$

Therefore,

$$\begin{aligned}
G(a+\epsilon) &= \left(1-3\frac{\epsilon}{a}\right)G(a)+3\frac{\epsilon}{a}\cdot\frac{25}{a^2}+o(\epsilon)\\
G(a+\epsilon)-G(a) &= -3\frac{\epsilon}{a}G(a)+3\frac{\epsilon}{a}\cdot\frac{25}{a^2}+o(\epsilon)\\
G'(a) &= -\frac{3}{a}G(a)+\frac{3}{a}\cdot\frac{25}{a^2}\\
a^3G'(a)+3a^2G(a) &= 75\\
\left[a^3G(a)\right]' &= 75\\
a^3G(a) &= 75a+K\\
G(5)=1 \implies K &= -250\\
G(a) &= \frac{75}{a^2}-\frac{250}{a^3}
\end{aligned}$$

So, finally, $G(10\sqrt{2})=0.2866$.