MASSACHUSETTS INSTITUTE OF TECHNOLOGY
1.203J/6.281J/13.665J/15.073J/16.76J/ESD.216J
Logistical and Transportation Planning Methods
Fall 2004

Assignment 6 Solutions

Prepared by Michael Metzger

1. Example 6.7

Let G be the graph under consideration with node set N. Let x be some point between nodes p and q. For all $x' \in G$, x is reachable from x' only through p or q. Therefore, the shortest path from x to x' is through either p or q. As a function of x, $d(x, x')$ is a piecewise linear function with slope ± 1 at each value of x. Similarly, it is easy to see that $d(x, p)$ and $d(x, q)$, as functions of x are linear with slope ± 1. Since $m(x) = \max_{j \in N} d(x, j)$ by definition, then $m(x)$ is the maximum of piecewise linear functions each having slope ± 1 for each value of x. So, $m(x)$ itself is also a piecewise linear function of x with slope at each point equal to ± 1.

Let $d_{(p,q)}(p, x)$ and $d_{(p,q)}(q, x)$ be the distances, along link (p, q) from x to p and from x to q, respectively. Because the slope of $m(x)$ is ± 1 for each point x on the link (p, q), as we move from $z = p$ to $z = x$, $m(z)$ can decrease by no more than $d_{(p,q)}(p, x)$. Similarly, as we move from $z = x$ to $z = q$, $m(z)$ can increase by no more than $d_{(p,q)}(x, q)$. That is,

$$\begin{aligned} m(x) &\geq \max\left[m(p) - d_{(p,q)}(p,x),\ m(q) - d_{(p,q)}(x,q))\right] \\ &\geq \frac{1}{2}\left[m(p) - d_{(p,q)}(p,x)\right] + \frac{1}{2}\left[m(q) - d_{(p,q)}(x,q)\right] \\ &= \frac{m(p) + m(q) - \ell(p,q)}{2} \end{aligned}$$

The second inequality follows since the maximum of two values is no smaller than their average.

2. LO Problem 6.6

(a)

$$\sum_{i\in N} P_i = \sum_{i\in N} \Big\{(\text{indegree of } i) - (\text{outdegree of } i)\Big\}$$
$$= \sum_{i\in N}(\text{indegree of } i) - \sum_{i\in N}(\text{outdegree of } i)$$
$$= \sum_{i\in N}\sum_{(k,i)\in A} 1 - \sum_{i\in N}\sum_{(i,k)\in A} 1 = 0\,.$$

Both of the last two sums count every directed arc of the network exactly once: the left-hand sum from the point of view of the tails and the right-hand sum from the point of view of the heads. Hence the difference of the two sums is zero (note that every arc (i,j) contributes exactly one to the outdegree of i and one to the indegree of j).

(b) In order to have a directed Euler tour, we must have $P_i' = 0$ for all nodes. Parallel to the undirected version, we add artificial arcs (i,j) between supply nodes $i \in S$ and demand nodes $j \in D$. Unlike the undirected version, where one additional arc was sufficient to make any odd node even, here it may be necessary to add many arcs to a node whose $|P_i|$ is large. In order to minimize the total length of arcs added, we construct $\sum_{i\in S} P_i$ minimum distance paths between the supply nodes and demand nodes. In order to ensure $P_i' = 0$ for all nodes, we require $\sum_{j\in D} x_{ij} = P_i$, $\forall i \in S$, which implies that

$$P_i' = P_i - \text{outdegree of new artificial arcs} = P_i - \sum_{j\in D} x_{ij} = 0\,.$$

Similarly, we require $\sum_{i \in S} x_{ij} = -P_j, \ \forall j \in D$, which ensures that

$$P'_j = P_j + \text{indegree of new artificial arcs} = P_j + \sum_{i \in S} x_{ij} = 0\,.$$

Here, x_{ij} represents the number of new artificial paths between nodes i and j. Since we now have $P'_i = 0, \ i \in S$ and $P'_j = 0, \ j \in D$, we can construct an Euler tour. It is certainly possible to use a link more than twice (See, for example, arcs (d, e) and (b, a) in the next part).

(c) **Step 1:** $S = \{b, d, g\}$ with $P_b = P_d = P_g = 1$, and $D = \{a, e\}$ with $P_a = -2, P_e = -1$. By inspection,

$$d(b,a) = 5, \quad d(b,e) = 17$$

$$d(d,a) = 14, \quad d(d,e) = 3$$

$$d(g,a) = 20, \quad d(g,e) = 9$$

Step 2:

$$\begin{aligned}
\text{minimize} \quad & z = 5x_{ba} + 17x_{be} + 14x_{da} + 3x_{de} + 20x_{ga} + 9x_{ge} \\
\text{subject to} \quad & x_{ba} + x_{be} = 1 \\
& x_{da} + x_{de} = 1 \\
& x_{ga} + x_{ge} = 1 \\
& x_{ba} + x_{da} + x_{ga} = 2 \\
& x_{be} + x_{de} + x_{ge} = 1 \\
& x_{ij} \in \{0, 1, 2, \dots\}
\end{aligned}$$

There are only three feasible integer solutions to this problem, so we enumerate these and check the value of the objective function in each case:

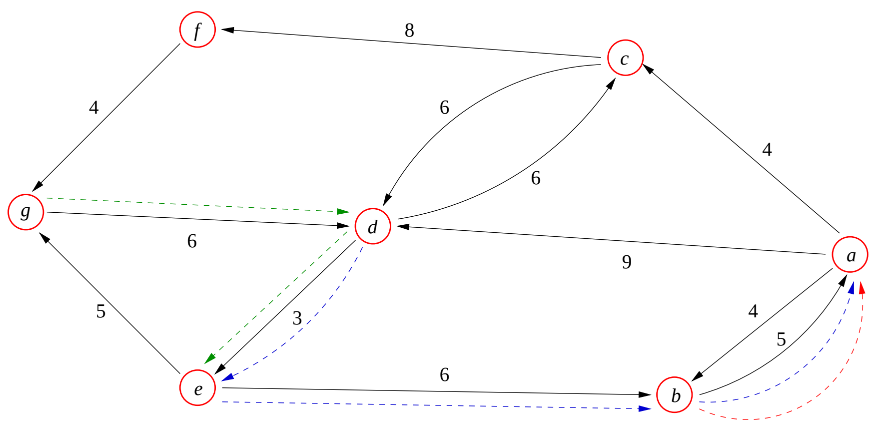
(1) $x_{ba} = 1, x_{be} = 0, x_{da} = 1, x_{de} = 0, x_{ga} = 0, x_{ge} = 1: \ z = 5 + 14 + 9 = 28$

(2) $x_{ba} = 0, x_{be} = 1, x_{da} = 1, x_{de} = 0, x_{ga} = 1, x_{ge} = 0: \ z = 17 + 14 + 20 = 51$

(3) $x_{ba} = 1, x_{be} = 0, x_{da} = 0, x_{de} = 1, x_{ga} = 1, x_{ge} = 0: \ z = 5 + 3 + 20 = 28$

Both solutions 1 and 3 are optimal. We choose the solution 1.

Step 3: We add paths from $b \to a$, $d \to a$ ($d \to e \to b \to a$), and $g \to e$ ($g \to d \to e$).



Step 4: $b \to a \to c \to d \to c \to f \to g \to d \to e \to g \to d \to e \to b \to a \to d \to e \to b \to a \to b$ is one possible tour.

(d) The suggested method forces us to traverse every undirected arc twice (once in each direction), which may not be optimal.

3. LO Problem 6.11

(a) Let G be an arbitrary tree.

We will prove the equality by proving $m(x) \leq m(x^*) + d(x, x^*)$ and $m(x) \geq m(x^*) + d(x, x^*)$.

- **Claim:** $m(x) \leq m(y) + d(x, y)$, $\forall x, y \in G$

  Pf: $\forall x, y, z \in G$, the shortest path from x to z is no worse than the shortest path from x to y followed by the shortest path from y to z. That is, using the function $d(\cdot, \cdot)$ as defined in the text,

$$d(x, z) \leq d(x, y) + d(y, z), \ \forall x, y, z \in G$$

  Let z be the point in the tree furthest from x. That is, z is s.t. $m(x) = d(x, z)$. Furthermore, note that $\forall y \in G$, $d(y, z) \leq m(y)$. Combining these relations, we obtain that, $\forall x, y \in G$.

$$\begin{aligned} m(x) &\leq d(x, y) + d(y, z) \\ &\leq d(x, y) + m(y) \end{aligned}$$

  Letting $y = x^*$, the yet unknown absolute center, we reach the desired result.

- **Claim:** $m(x) \geq m(x^*) + d(x, x^*)$

  Pf: We must first show

  - **Sub-claim:** $\exists p, p' \in G$ s.t. $p \neq p'$, $m(x^*) = d(x^*, p) = d(x^*, p')$, and x^* lies on the unique path from p to p'.

    Pf of Sub-claim: Consider the case where x^* does not coincide with a node. Let q and q' be the nodes s.t. x^* is contained on the link (q, q'). For any $r \in N$, the set of nodes in the network, the path from r to x^* must pass through either q or q'. Therefore, we can partition N into N_q and $N_{q'}$, the set of nodes accessible to x^* via a path through q

and q', respectively. N_q and $N_{q'}$ are disjoint since the path between any two nodes in a tree is unique. Therefore, the tree cannot contain both a path from r to x^* via q and one from r to x^* via q' (recall that a path is a "journey" through the network that never covers the same point twice).

Note that $\forall r \in N$, $d(r, x^*) \leq m(x^*)$, by the definition of $m(x^*)$. Let p' be the node furthest from x^*, i.e. $d(p', x^*) = m(x^*)$. Without loss of generality, suppose that $p' \in N_{q'}$. Suppose there is no $r \in N_q$ s.t. $d(r, x^*) = m(x^*)$. That is, $\forall r \in N_q$, $d(r, x^*) < m(x^*)$. Accordingly, $\exists \epsilon > 0$ s.t. $\forall r \in N_q$, $d(r, x^*) \leq m(x^*) - \epsilon$. Now consider the point y which is also on the link (q, q') but is $\frac{\epsilon}{2}$ units closer to q' than is x^*. Then, $\forall r \in N_q$, $d(r, y) = d(r, x^*) + \frac{\epsilon}{2} < m(x^*)$ and $\forall r \in N_{q'}$, $d(r, y) = d(r, x^*) - \frac{\epsilon}{2} < m(x^*)$. Since $m(y) = \max_{j \in N} d(j, y)$, then $m(y) < m(x^*)$. This yields a contradiction, since x^* cannot be the absolute center. So, $\exists p \in N_q$ s.t. $d(p, x^*) = m(x^*)$. Since $p \in N_q$ and $p' \in N_{q'}$, by the definitions of these sets, the path from p to p' must cover the link (q, q') and therefore passes through x^*.

The case of x^* coinciding with a node is completely analogous but messier, so we will leave it out and assume the claim proved.

– **Sub-Claim:** $\forall x \in G$, x^* lies on the path between x and p or on the path between x and p'

  Pf: Suppose not. Then the path from p to x to p' does not go through x^*. However, we already said that x^* must lie on the path between p and p'. Since a tree contains a unique path between any two nodes, this yields a contradiction.

Without loss of generality, suppose that x^* lies on the path from x to p'. Then

$$\begin{aligned} m(x) &\geq d(x, p') \\ &= d(x, x^*) + d(x^*, p') \\ &= d(x, x^*) + m(x^*) \end{aligned}$$

Since $m(x) \geq m(x^*) + d(x, x^*)$ and $m(x) \leq m(x^*) + d(x, x^*)$ $\forall x \in G$, we have that $m(x) = m(x^*) + d(x, x^*)$ $\forall x \in G$.

(b) Now consider any $x \in G$ and let e_s be the farthest point from x. That is, $m(x) = d(x, e_s)$.

- **Claim:** x^* lies on the path from x to e_s.

  Pf: Suppose not, then

$$\begin{aligned} d(x, x^*) + d(x^*, e_s) &> d(x, e_s) \\ &= m(x) \\ &= m(x^*) + d(x^*, x), \text{ from part (a)} \\ m(x^*) &< d(x^*, e_s), \text{ by subtracting } d(x^*, x) \text{ from each side} \\ &\leq m(x^*), \text{ by definition of } m(x^*) \end{aligned}$$

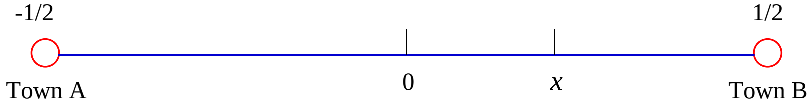
Contradiction, since $m(x^*) < m(x^*)$ is impossible.

Now, using the fact that x^* lies on the path from x to e_s, we have that

$$\begin{aligned} m(e_s) &= m(x^*) + d(x^*, e_s), \text{ from part (a)} \\ &= m(x^*) + (d(e_s, x) - d(x^*, x)) \\ &= m(x^*) + m(x) - d(x^*, x), \text{ since } d(e_s, x) = m(x) \\ &= m(x^*) + (m(x^*) + d(x^*, x)) - d(x^*, x), \text{ from part (a)} \\ &= 2m(x^*) \end{aligned}$$

If we repeat the proof of the Claim using e_s in place of x, we know that x^* must lie on the path from e_s to the furthest point from e_s, namely e_t. Therefore, $d(x^*, e_s) = m(x^*) = \frac{1}{2}m(e_s) = \frac{1}{2}d(e_s, e_t)$. The first equality holds since, if there were a vertex further from x^* than e_s in the direction of e_s, then it would also be further from e_t than e_s (contradiction). So, x^* is the midpoint between e_s and e_t.

4. LO Problem 6.17



(a) Let S denote the service time (travel time + on-scene time) for a random patient. Since the travel speed v is 1 and the on-scene time τ is 1, we have the following service time PMF:

$$p_S(s) = \begin{cases} f_A, & s = 2(0.5 + x) + 1 \\ f_B, & s = 2(0.5 - x) + 1 \end{cases}$$

From the PMF, we obtain

$$E[S] = f_A\big(2(0.5+x)+1\big) + f_B\,\big(2(0.5-x)+1\big) = 2(f_A - f_B)x + 2\,,$$

$$E[S^2] = f_A\big(2(0.5+x)+1\big)^2 + f_B\big(2(0.5-x)+1\big)^2 = 4x^2 + 8(f_A - f_B)x + 4\,.$$

Let R be the response time to a random patient. Then $E[R] = W_q + E[S]$, where W_q is the average waiting time in the queue. Using the formula for W_q of the M/G/1 queueing system,

$$\begin{aligned} E[R] &= \frac{\lambda(E[S]^2 + \sigma_S^2)}{2(1-\lambda E[S])} + E[S] = \frac{\lambda E[S^2]}{2(1-\lambda E[S])} + E[S] \\ &= \frac{\frac{1}{4}\big\{4x^2 + 8(f_A - f_B)x + 4\big\}}{2\big\{1 - \frac{1}{4}(2(f_A - f_B)x + 2)\big\}} + 2(f_A - f_B)x + 2 \\ &= \frac{x^2 + 2(f_A - f_B)x + 1}{1 - (f_A - f_B)x} + 2(f_A - f_B)x + 2\,. \end{aligned}$$

(b) If $f_A = f_B = \frac{1}{2}$, $E[R] = x^2 + 3$. The value of x that minimizes $E[R]$ is 0. Hence the optimal location of the hospital is the mid-point between the two towns.

(c) The answer in (b) does not agree with Hakimi's theorem. If Hakimi's theorem held, Town A or Town B would be the optimal location. In fact, locating the hospital at Town A or Town B maximizes $E[R]$. Note that $E[R]$ is a convex function of x. Hakimi's theorem holds when a disutility function to be minimized is concave (or equivalently a utility function to be maximized is convex. Refer to the theorem in Section 6.5.3 of the textbook). We have also seen in class that the Stochastic Queue Median (SQM) is not at nodes in general.

(d) For the system to reach steady-state, $\rho = \lambda E[S]$ should be less than 1. Since $E[S]$ is a function of x, the question of whether steady-state is reached depends on the location of the hospital.

(e) Since no queue is allowed, $W_q = 0$. Hence we want to find the value of x that minimizes

$$E[R] = E[S] = 2(f_A - f_B)x + 2 = 1.2x + 2\,.$$

$E[R]$ is minimized when $x = -0.5$, so the hospital should be located at Town A. This exemplifies that the Stochastic Loss Median (SLM) is a Hakimi median.

(f) We can set up the following linear optimization model for this question:

$$\begin{aligned} \text{minimize} \quad & 1.2x + 2 \\ \text{subject to} \quad & 2(0.5 + x) + 1 \le 2.6 \\ & 2(0.5 - x) + 1 \le 2.6 \\ & -0.5 \le x \le 0.5 \end{aligned}$$

The first two constraints yield $-0.3 \le x \le 0.3$ (The third constraint is redundant). Since the objective function is increasing, the optimal value of x is -0.3.

5.

Question 1: we know, to begin with, that TD, the MST of the points in D has an even number of odd–degree nodes (just like any undirected network). Let us denote as v the point in D which is closest to s (i.e., v is the point in D with which s is connected in the overall tree T). We now have two cases:

> (i) The point v had even degree in TD. In this case, the addition of the edge (s,t) will make v a node of odd degree in T. This increases the number of odd–degree nodes in the "D part" of the tree T by 1, i.e., we have an odd number of odd–degree nodes in RÇD.
>
> (ii) The point v has odd degree in TD. In this case, the addition of the edge (s,t) will make v a node of even degree in T. This decreases the number of odd–degree nodes in the "D part" of the tree T by 1, i.e., we have again an odd number of odd–degree nodes in RÇD.

An entirely parallel argument can be made about RÇP.

Question 2:

(a) This part is obvious. Since H adds one more incidence to all the odd–degree nodes in T, the graph G (=TÈH) has no nodes of odd–degree (and is connected) so it has an Euler tour.

(b) The key observation here is that, because of the large additional cost K associated with each pairing of a point in D with a point in P, there will be only one pairing of an odd–degree point in D (call it z) with an odd–degree point in P (call it w) in the optimal matching. (Note that from Question 1 we know that there will be one "left–over" odd–degree point from D and one "left–over" odd–degree point in P, after we have finished the pairwise matching of odd–degree points in D with one another and of odd–degree points in P with one another; please also note that, by construction, s will always have a degree of 2 in T.)

Thus we can begin at s, find an Euler path from s to z that visits all the points in D at least once, then use the link (z,w) to go to the points in P and then find an Euler path from w to s that visits all the points in P at least once.

6.

(a) We need a minimal length pair-wise matching of nodes of odd degree, in order to make a new network or graph having all nodes of even degree. Then we can construct an Euler tour. By inspection the matching corresponds to appending to the original network duplicate links for all interior bridges. This is because the bridges are by far the shortest length links that we can use to create an augmented network having all even degree nodes. In practice, this means that the jogger, when approaching a bridge that he has not yet jogged across, would jog across it and then immediately make a U turn and jog back across it. (No need to match the two bridges on the two far ends of the total jogging route, as they have nodes of even degree; all others have nodes of odd degree.) The total length of the jog is then 18.27 + 1.57 = 19.84 miles.

<u>Note 1:</u> One student came up with another way to implement the Euler tour, one that is much less boring from a jogger's point of view. Start at the Science Museum north

and jog south along Science Museum land bridge; jog to Longfellow and jog across it; jog to Mass Ave Bridge and jog across it,…continue this alternating path until jogger reaches south end of Watertown Sq. Bridge and jogs across it; now return on the Cambridge side and again jog across each bridge you come across. It works! Draw a picture!

<u>Note 2:</u> Another student noted quite correctly that we did not explicitly ask for an Euler *tour* in part (a), only an Euler *path*. Thus, allowing for Mike Jogger to end up at a different location than his starting location, we do not have to add the longest bridge – the Longfellow Bridge – to our augmented graph. We can allow in the augmented graph two nodes of odd degree. The jogger can start at one end of the Longfellow Bridge and finish his jogging path at the other end. Full credit was given for both interpretations.

(b) There are 122 = 144 equally likely jogging tours. Note that tour (3,7) for instance is different from tour (7,3); the same path is followed around the two selected bridges, but in reverse directions. On any given day tour *i* occurs with probability 1/144. To construct the probability law for the total jogging distance on a random day, we just compute the jogging length (in miles) for each of the possible 144 tours and assign the probability 1/144 to each. Note that the distance for any tour must include the extra distance, if any, required for the jogger to get to and from the Operations Research Center at MIT to the closer of the two bridges in his tour. We can express the result of combining 144 such calculations either as a probability mass function or as a cumulative distribution function. Suppose the total jogging distance for tour (*i,j*) is *dij*. Then the expected jogging distance on a random day is

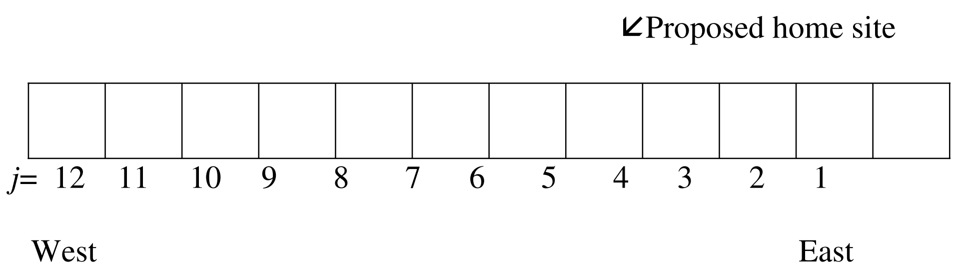
$$E[D] = \sum_{i=1}^{12}\sum_{j=1}^{12} d_{ij} / 144$$

(c) Each jogging tour has a minimal jogging distance corresponding to the sum of the two bridge lengths plus the two land distances between the two bridges crossed. If the home location of the jogger on the Cambridge side of the river is between the two bridges crossed, then that location adds zero additional mileage to the route.

However, if that home location is not between the two bridges, then there is a total ‘deadheading’ distance equal to twice the distance from his home location to the closer of the two bridges. This is extra distance he has to jog to get to and return from the cyclic route connecting the two bridges selected for that day. To minimize the expected distance jogged per day, the jogger has to minimize (twice) the expected deadheading distance.

Suppose the jogger’s proposed home location is west of bridge *j* and east of bridge *j*+1, *j* =1,2,…, 11, with the bridges sequentially numbered from 1 to 12, starting with the Science Museum land bridge (at *j* = 1). In the figure below, *j* = 5.

Cambridge Side of the River



That is, the proposed home location is on the Cambridge side between bridges *j* and *j*+1. Then there are *j2* deadheading routes east of the home location and (12-*j*)2 deadheading routes west of the home location. Why? The remaining 2*j*(12-*j*) tours have no deadheading distance. Why? These observations reduce the problem to a 1-median problem on a straight line. At any proposed home location for the jogger, there are 'nodal weights' totaling *j2* /144 to the east and (12-*j*)2/144 to the west. For instance if *j2* /144 > (12-*j*)2/144, then the mean daily jogging distance is reduced in a linear manner as one moves the jogger's home location closer to bridge *j* and farther from bridge *j*+1. One seeks a home location for the jogger at which the 'weights' pulling left and right are equal. For if they are not equal, one reduces mean deadheading distance by moving the home location of the jogger in the direction having greater total weight. The optimal balance occurs in this problem at any point between bridge 6 (Western Avenue Bridge) and bridge 7 (Weeks Footbridge), including the two nodes corresponding to the respective end points of those bridges. At these points the two sets of weights are equal, each being 36/144. We recall in the one median that we may have non-nodal optimal locations (in addition to nodal ones) if the weights pulling in each direction are equal. If the total number *N* of bridges in this problem had been an odd number rather than an even number, then the optimal home location would be at a node equal to the bridge number (*N*+1)/2. For instance, if there had been *N* = 13 rather then 12 bridges in this problem then the optimal location would be at the node corresponding to bridge (13 + 1)/2 = 7 (Weeks Footbridge). Just as in the regular 1-median location problem, one does not risk missing an optimal solution to the problem by examining possible home locations solely at the nodes. This is true whether there is an even or an odd number of bridges in the problem.

7.

(a) As suggested, we prove the result by contradiction. Suppose the solution is at node $y \in S$ and that the set of nodes T contains no solution to the 1-median problem.

$$\begin{aligned}
J(y) &= \sum_{j \in T} h(j)d(y,j) + \sum_{j \in T} h(j)d(y,j) \\
&= \sum_{j \in T} h(j)\left(d(y,t) + d(t,j)\right) + \sum_{j \in S} h(j)d(y,j) \\
&= \sum_{j \in T} h(j)d(t,j) + H(T)d(y,t) + \sum_{j \in S} h(j)d(y,j) \\
&\geq \sum_{j \in T} h(j)d(t,j) + H(S)d(y,t) + \sum_{j \in S} h(j)d(y,j) \\
&= \sum_{j \in T} h(j)d(t,j) + \sum_{j \in S} h(j)\left(d(j,y) + d(y,t)\right) \\
&= \sum_{j \in T} h(j)d(t,j) + \sum_{j \in S} h(j)d(t,j) \\
&= J(t)
\end{aligned}$$

This yields a contradiction, since we would do no worse by locating the 1-median at $t \in T$.

(b) From part (a), we know that an optimal solution exists in T. $\forall y \in T$,

$$\begin{aligned}
J(y) &= \sum_{j \in S} h(j)d(y,j) + \sum_{j \in T} h(j)d(y,j) \\
&= \sum_{j \in S} h(j)d(y,j) + h(t)d(y,t) + \sum_{j \in T-t} h(j)d(y,j) \\
&= \sum_{j \in S} h(j)\left(d(y,t) + d(t,j)\right) + h(t)d(y,t) + \sum_{j \in T-t} h(j)d(y,j) \\
&= \sum_{j \in S} h(j)d(t,j) + H(S)d(y,t) + h(t)d(y,t) + \sum_{j \in T-t} h(j)d(y,j) \\
&= K + \left(H(S) + h(t)\right)d(y,t) + \sum_{j \in T-t} h(j)d(y,j) \\
&= K + \tilde{J}(y)
\end{aligned}$$

where $K = \sum_{j \in S} h(j)d(t, j)$, a constant with respect to y, and $\tilde{J}(\cdot)$ is the objective function for the 1-median problem on $G'(T, A_t)$ with the weight of node t given by $H(S) + h(t)$.

(c) Isthmus edge (g, i) separates the network into two distinct subnetworks with node sets S_1 and T_1, where

$$S_1 = \{a, b, c, d, e, f, g, h\}$$
$$T_1 = \{i, j, k, l, m, n, o, p, q\}$$

$H(S_1) = 32$ and $H(T_1) = 41$. Therefore, an optimal solution must be one of the nodes in T_1.

We now disregard the portion of the network involving nodes in S_1 and increase the weight at i to $h(i) + H(S) = 5 + 32 = 37$. Consider isthmus edge (i, j), which divides the new graph into two distinct subnetworks with $S_2 = \{j, k, l, m\}$ and $T_2 = \{i, n, o, p, q\}$. Clearly $H(S_2) < H(T_2)$ (remember that $h(i)$ is now 37). So, we can disregard the portion of the new network involving nodes in S_2. And again, we must increase the weight at i by $H(S_2)$. So, the weight at node i becomes $37 + H(S_2) = 37 + 17 = 54$.

The new network consists only of nodes i, n, o, p, q and edges between pairs of nodes from this set. Now consider isthmus edge (i, n) which divides the new graph into node sets $S_3 = \{i\}$ and $T_3 = \{n, o, p, q\}$. $H(S_3) = 54$ and $H(T_3) = 19$. Therefore, an optimal solution to the 1-median problem is to locate at node i.