Urban OR-Spatial Poisson Example
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Suppose that emergency vehicles are distributed over a region under a spatial Poisson process with parameter λ. Let v_i be the straight-line distance from point P to the i^{th} nearest emergency vehicle. We seek $P(v_2 > 2v_1)$, which is the chance that the second-nearest vehicle is more than twice as far from P as the nearest one.

As a start, suppose that $v_2 = Q$. Then, we know that there is exactly one vehicle in the circle of radius Q centered at P. (Right?) Under the Poisson assumption, that vehicle is equally likely to be anywhere within that circle.

The event $v_2 > 2v_1$ requires in this instance that $v_1 < Q/2$. We know that the vehicle nearest P is uniformly distributed over a circle with radius Q, and need the probability that it falls within the circle of radius Q/2 centered at P. That probability is simply 1/4, because the smaller circle has half the radius of the larger one and thus only one quarter the area. To put it another way, 25% of the points within distance Q of P are within distance Q/2.

This result is conditioned on the fact that $v_2 = Q$, but it is obvious that the same outcome would arise whatever the value of v_2. Thus, $P(v_2 > 2v_1) = 0.25$. The chances are 75% that the second nearest vehicle is less than twice as far from P as the nearest one.

What is happening here is that the distance from P to the k^{th} nearest vehicle does not rise linearly with k. On average, the *area* we have to search to find the vehicle second nearest to P is twice as large as that to find the nearest. But area varies with the *square* of distance from P, so doubling the area typically involves a rise in distance tied to the square root of 2, which is 1.41.