Lecture Notes on Fluid Dynamics
(1.63J/2.21J)
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6.5 Geothermal Plume

6-5g-plume-L.tex

R..A.Wooding, (1963), *J Fluid Mech.* 15, 527-544.
C. S. Yih, (1965), *Dynamics of Nonhomogeneous Fluids*, Macmillan.
D. A. Nield and A. Bejan, (1992), *Convection in Porous Media.* Springer-Verlag.

Consider a steady, two dimensional plume due to a source of intense heat in a porous medium. From Darcy's law:

$$\frac{\mu}{k}u = -\frac{\partial p}{\partial x} \tag{6.5.1}$$

where k denotes the permeability, and

$$\frac{\mu}{k}w = -\frac{\partial p}{\partial z} - \rho g \tag{6.5.2}$$

These are the momentum equations for slow motion in porous medium. Mass conservation requires

$$u_x + w_z = 0 \tag{6.5.3}$$

Energy conservation requires

$$u\frac{\partial T}{\partial x} + w\frac{\partial T}{\partial z} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2}\right) \tag{6.5.4}$$

where

$$\alpha = \frac{K}{\rho_0 C} \tag{6.5.5}$$

denotes the thermal difusivity.

Equation of state:

$$\rho = \rho_0\left(1 - \beta(T - T_0)\right) \tag{6.5.6}$$

Consider th flow induced by a strong heat source. Let

$$T - T_0 = T', \quad p = p_o + p'$$

where p_0 is the hydrostatic pressure satisfying

$$-\frac{\partial p_0}{\partial z} - \rho_0 g = 0.$$

Eqn. (6.5.2) can be written

$$\frac{\mu}{k}w = -\frac{\partial p'}{\partial z} + g\rho_0\beta T'. \tag{6.5.7}$$

6.5.1 Boundary layer approximation

Eliminating p' from Eqns. (6.5.7) and (6.5.1), we get

$$\frac{\mu}{k}(w_x - u_z) = g\rho_0\beta T'_x.$$

Let ψ be the stream funciton such that

$$u = \psi_z, \quad w = -\psi_x$$

then

$$\psi_{xx} + \psi_{zz} = -\frac{g\rho_0\beta k}{\mu}T'_x \tag{6.5.8}$$

For an intense heat source, we expect the plume to be narrow and tall. Let us apply the boundary layer approximation and check its realm of validity later,

$$u \ll w, \qquad \frac{\partial}{\partial x} \gg \frac{\partial}{\partial z}.$$

hence

$$\psi_{xx} \cong -\frac{\rho_0\beta k}{\mu}T'_x$$

or

$$\psi_x \cong -\frac{g\rho_0\beta k}{\mu}T', \tag{6.5.9}$$

which is the same as ignoring $\partial p'/\partial z$ in Eqn. (6.5.7).

This can be confirmed since $u \ll w$ $\partial p'/\partial x \approx 0$, p' inside the plume is the same as that outside the plume. But

$$\frac{\partial p'}{\partial z} = 0$$

outside the plume, hence $\partial p'/\partial z \approx 0$ inside as well.

Applying the B.L. approximation to Eqn. (6.5.4)

$$uT'_x + wT'_z = \alpha T'_{xx} \tag{6.5.10}$$

Using the continuity equation we get

$$(uT')_x + (wT')_z = \alpha T'_{xx}.$$

Integrating across the plume,

$$\frac{\partial}{\partial z}\int_{-\infty}^{\infty} wT'\,dx = 0 \tag{6.5.11}$$

since $T' = 0$ outside the plume. It follows that

$$\rho_o C\int_{-\infty}^{\infty} wT'\,dx = -\rho_0 C\int_{-\infty}^{\infty} \psi_x T'\,dx = Q = \text{constant}. \tag{6.5.12}$$

6.5.2 Normalization

Let us take

$$x = B\bar{x}, \quad z = H\bar{z}, \quad u = \frac{WB}{H}\bar{u}, \quad w = W\bar{w}, \quad T' \to \Delta T\theta \tag{6.5.13}$$

where $H, B, \Delta T$ and W are to be determined to get maximun simplicity. We then get from the momentum equation,

$$\bar{w} = \bar{\psi}_{\bar{x}} = -\frac{g\rho_0\beta\Delta T}{\mu W}\theta,$$

from the energy equation,

$$\bar{u}\theta_{\bar{x}} + \bar{w}\theta_{\bar{z}} = \frac{\alpha H}{WB^2}\theta_{\bar{x}\bar{x}},$$

and from the total flux condition,

$$\rho_0 CWB\Delta \int_{-\infty}^{\infty} \bar{w}\theta d\bar{x} = Q$$

Let us choose

$$\frac{g\rho_0\beta\Delta T}{\mu W} = 1 \tag{6.5.14}$$

$$\frac{\alpha H}{WB^2} = 1 \tag{6.5.15}$$

and

$$\rho_0 CWB\Delta T = Q, \tag{6.5.16}$$

which gives three relations among four scales, $B, H, W, \Delta T$. Then

$$\bar{w} = \bar{\psi}_{\bar{x}} = -\theta, \tag{6.5.17}$$

from the energy equation,

$$\bar{u}\theta_{\bar{x}} + \bar{w}\theta_{\bar{z}} = \theta_{\bar{x}\bar{x}}, \tag{6.5.18}$$

and from the total flux condition,

$$\int_{-\infty}^{\infty} \bar{w}\theta d\bar{x} = 1 \tag{6.5.19}$$

In addition we require that

$$w(\pm\infty, z) = 0, \quad \theta(\pm\infty, z) = 0 \tag{6.5.20}$$

$$u(0, z) = \frac{\partial w(0, z)}{\partial x} = 0, \quad x = 0. \tag{6.5.21}$$

From here on we omit overhead bars in all dimensionless equations for brevity.

6.5.3 Similarity solution

Now let

$$x = \lambda^a x^* \quad z = \lambda^b z^* \quad \psi = \lambda^c \psi^* \quad \theta = \lambda^d \theta^*.$$

From Eqn. (6.5.17)

$$\lambda^{c-a} \left(\frac{\partial \psi^*}{\partial x^*} \right) = -\lambda^d \theta^*.$$

For invariance we require,

$$c - a = d. \tag{6.5.22}$$

From (6.5.19)

$$-\int \frac{\partial \psi^*}{\partial x^*} \, dx^* \; \lambda^{c-a+a+d} = 1.$$

therefore,

$$a + d = 0. \tag{6.5.23}$$

From Eqn. (6.5.18)

$$\lambda^{c+d-a-b} = \lambda^{d-2a}.$$

implying,

$$c + a - b = 0. \tag{6.5.24}$$

Finally

$$c = \frac{a}{2}, \quad d = -\frac{a}{2}, \quad b = \frac{3}{2} a.$$

In view of these we introduce the following similarity variables,

$$\eta = \frac{x}{z^{2/3}}, \quad \psi = z^{1/3} f(\eta), \quad \theta = z^{-1/3} \, h(\eta). \tag{6.5.25}$$

Note that at the center line $\eta = 0$

$$w = -\psi_x \propto z^{1/3} f'(0)(-) z^{-2/3} \sim z^{-1/3} f'(0) \sim z^{-1/3} \tag{6.5.26}$$

$$\theta \propto z^{-1/3} h(0) \tag{6.5.27}$$

and

$$b \propto z^{2/3} \tag{6.5.28}$$

Thus the velocity and temperature along the centerline decay as $z^{-1/3}$ and the plume width grows as $z^{2/3}$.

Substituting these into Eqns. (6.5.17) and (6.5.18), we get, after some algebra

$$-\frac{df}{d\eta} = h \tag{6.5.29}$$

and

$$\frac{d}{d\eta}(fh) = 3 \frac{d^2 h}{d\eta^2}. \tag{6.5.30}$$

The boundary conditions are,

$$\begin{aligned} f &= 0 \qquad (\psi = 0) \\ f''(0) &= 0, \quad (w(0,z) = w_{max}) \\ f(\pm\infty),\ f'(\pm\infty) &= 0 \\ h(\pm\infty) &= 0. \end{aligned}$$

Integrating Eqn. (6.5.30), we get

$$fh = 3h'.$$

Using Eqn. (6.5.29), we get

$$ff' = 3f''.$$

Integrating again, we get

$$-6f' = f_0^2 - f^2$$

where $f_0 = f_{\text{max}}$. Let $f = -f_0 F$, then

$$f_0(1 - F^2) = 6F', \text{or} \quad \frac{dF}{1 - F^2} = \frac{f_0 d\eta}{6}$$

which can be integrated to give

$$\frac{f_0\eta}{6} = \frac{1}{2}\ln\frac{1+F}{1-F}$$

Thus

$$\left(\frac{1+F}{1-F}\right)^{1/2} = e^{f_0\eta/6}$$

or

$$\left(\frac{1+F}{1-F}\right) = e^{f_0\eta/3}$$

Solving for F, we get

$$F = \frac{e^{f_0/3} - 1}{e^{f_0/3} + 1} = \tanh\frac{f_0\eta}{6} \tag{6.5.31}$$

What is f_0? Let us use Eqn. (6.5.29)

$$-\int_{-\infty}^{\infty} \frac{df}{d\eta}\, h\, d\eta = \int_{-\infty}^{\infty} (f')^2 d\eta = 1$$

since

$$f' = -f_0 F' = -\frac{f_0^2}{6}\text{sech}^2\frac{f_0\eta}{6}$$

and

$$h = -f'.$$

Therefore,

$$\left(\frac{f_0^2}{6}\right)^2 \int_{-\infty}^{\infty} \operatorname{sech}^4\left(\frac{f_0\eta}{6}\right) d\eta = \frac{f_0^3}{6}\int_{-\infty}^{\infty} \operatorname{sech}^4 \zeta d\zeta = 1.$$

Since

$$\int_{-\infty}^{\infty} \operatorname{sech}^4 z dz = 4/3.$$

we get f_0!

$$f_0 = \left(\frac{9}{2}\right)^{1/3} \tag{6.5.32}$$

The solution is

$$f = \left(\frac{9}{2}\right)^{1/3} \tanh\left(\frac{9}{2}\right)^{1/3} \frac{\eta}{6} \tag{6.5.33}$$

and

$$h = -f' = -\left(\frac{9}{2}\right)^{2/3} \operatorname{sech}^2\left(\frac{9}{2}\right)^{1/3} \frac{\eta}{6} \tag{6.5.34}$$

Computed results are given in Figures.

Remark Checking the boundary layer approximation.

$$\frac{\partial^2\psi}{\partial x^2} \sim z^{-1}, \quad \frac{\partial^2\psi}{\partial z^2} \sim z^{-5/3}$$

$$\frac{\partial^2 T'}{\partial x^2} \sim z^{-5/3}, \quad \frac{\partial^2 T'}{\partial z^2} \sim z^{-7/3}$$

hence for large z, B. L. approximation is good.

6.5.4 Return to physcial coordinates

Start from

$$\eta = \frac{\bar{x}}{\bar{z}^{2/3}} \tag{6.5.35}$$

$$\frac{\bar{\psi}}{\bar{z}^{1/3}} = f(\eta) \tag{6.5.36}$$

$$\bar{z}^{1/3}\theta = h(\eta) \tag{6.5.37}$$

Then

$$\eta = \frac{x/B}{(z/H)^{2/3}} = \left(\frac{H^{2/3}}{B}\right)\left(\frac{x}{z^{2/3}}\right) \tag{6.5.38}$$

By eliminating H and ΔT from(6.5.35) and (6.5.37), we get

$$W = \sqrt{\frac{Qg\beta}{CB}}$$

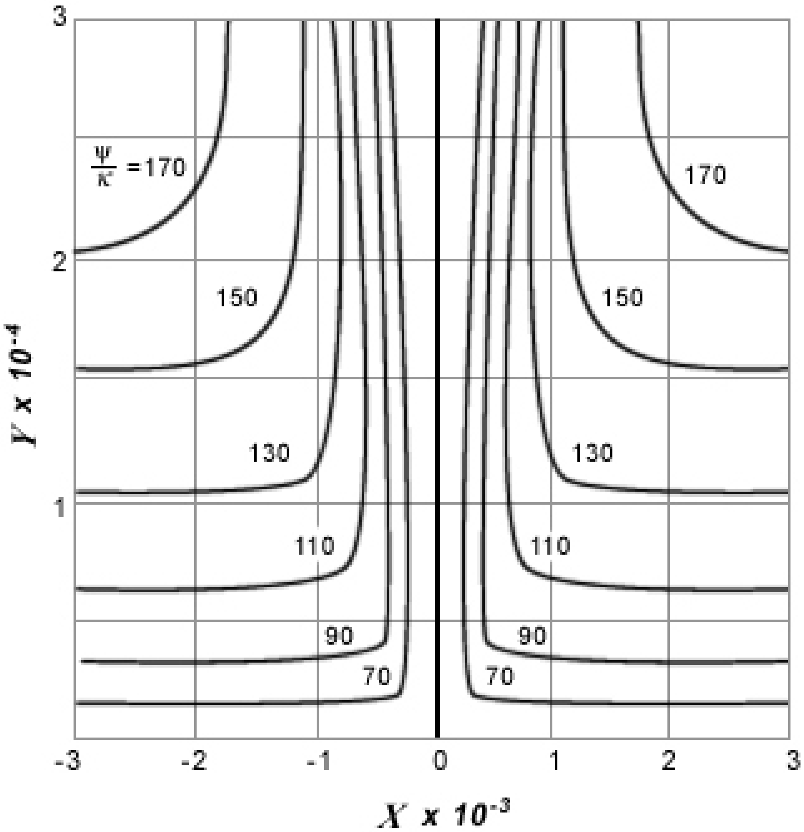

Figure 6.5.1: *Theoretical solution for a geothermal plume due to Yih.*
(Adapted from Yih, Dynamics of Nonhomogeneous Fluids,1965).

From (6.5.36), we get

$$\frac{H}{B^2} = \frac{W}{\alpha} = \frac{1}{\alpha}\sqrt{\frac{Qg\beta}{CB}}$$

It follows that

$$\frac{H}{B^{3/2}} = \frac{1}{\alpha}\sqrt{\frac{Qg\beta}{C}} \tag{6.5.39}$$

Now

$$\frac{\bar{\psi}}{\bar{z}^{1/3}} = \frac{\psi}{WB}\left(\frac{z}{H}\right)^{-1/3} = \left(\frac{H^{1/3}}{WB}\right)\left(\frac{\psi}{z^{1/3}}\right) \tag{6.5.40}$$