Massachusetts Institute of Technology

Quantum Mechanics I (8.04) Spring 2006

Solutions to Practice Exam 1

1. **Short problems.** (30 points)

   (a) i. (5 points) Substantially stronger backscattering of α-particles from a Au foil was observed than expected if the α-particles were scattering off light electrons. This implied that the positive charge and mass were concentrated in a small region of space.

   ii. (5 points) Rutherford formula for positive nuclei breaks down when α-particle penetrates nucleus.

   Closest approach for backscattering (K.E.=0 at turning point),

$$E = \frac{ZZ'q^2}{4\pi\varepsilon_0 R} \ . \tag{1}$$

   where $R = 10^{-15}$ m size of nucleus.

$$E = \frac{2.47 \cdot (1.6 \times 10^{-19}\ \mathrm{C})^2}{4\pi \cdot 8.85 \times 19^{-12}\ \mathrm{C}^2/(\mathrm{Nm}^2) \cdot 10^{-15}\ \mathrm{m}} = 2.2 \times 10^{-11}\ \mathrm{J} = 135\ \mathrm{MeV} \ . \tag{2}$$

   (b) (10 points) Largest shift corresponds to largest energy loss of photon due to momentum transfer onto electron. Largest momentum transfer for backscattering, $\theta = \pi$,

$$E' = \frac{1}{2}E \ , \tag{3}$$

   where E and E' are photon energies before and after scattering.

$$\begin{aligned}
\lambda' &= 2\lambda && (4)\\
\lambda' - \lambda &= \frac{h}{mc}(1-\cos\pi) = \frac{2h}{mc} && (5)\\
\lambda' - \lambda &= 2\lambda - \lambda = \lambda = \frac{2h}{mc} && (6)\\
E &= \frac{hc}{\lambda} = \frac{hc}{2h}mc = \frac{1}{2}mc^2 = \frac{1}{2}\cdot 511\ \mathrm{MeV} = 256\ \mathrm{MeV} \ . && (7)
\end{aligned}$$

   (c) (10 points)

$$eU = \frac{hc}{\lambda} \Longrightarrow U = \frac{hc}{e\lambda} = 12.4\ \mathrm{eV} \tag{8}$$

   Fractional correction: $\frac{5\ \mathrm{eV}}{12.4\ \mathrm{keV}} = 4 \times 10^{-4}$.

(a) (5 points) The normalization condition requires

$$\int_{-\infty}^{\infty} |\psi|^2 \, dx = 1 \tag{27}$$

which in this case becomes

$$\int_{-b}^{3b} |A|^2 \, dx = 1 = 4b\,|A|^2 \qquad \Longrightarrow \qquad A = \frac{1}{2\sqrt{b}} \tag{28}$$

(b) (5 points) Probability of finding the particle in the $x \in [0, b]$ region is given by

$$\int_{0}^{b} |\psi|^2 \, dx = \int_{0}^{b} \frac{1}{4b} = \frac{1}{4} \tag{29}$$

(c) (10 points) The expectation values are

$$\langle x \rangle = \int_{-\infty}^{\infty} x\,|\psi|^2 \, dx = \int_{-b}^{3b} x \frac{1}{4b} \, dx = b \tag{30}$$

$$\left\langle x^2 \right\rangle = \int_{-\infty}^{\infty} x^2\,|\psi|^2 \, dx = \int_{-b}^{3b} x^2 \frac{1}{4b} \, dx = \frac{7}{3} b^2 \tag{31}$$

d) Momentum probability density $= |\phi(p)|^2$

$$\phi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dx\, \psi(x)\, e^{-ipx/\hbar}$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \int_{-b}^{3b} dx\, A\, e^{-ipx/\hbar}$$

$$= \frac{i\hbar A}{\sqrt{2\pi\hbar}\, p} \left[e^{-ipx/\hbar} \right]_{-b}^{3b}$$

$$= \frac{iA}{p} \sqrt{\frac{\hbar}{2\pi}} \left[e^{-3ipb/\hbar} - e^{+ipb/\hbar} \right]$$

$$= 2\frac{A}{p} \sqrt{\frac{\hbar}{2\pi}}\, e^{-ipb/\hbar} \underbrace{\frac{1}{2i} \left[e^{+2ipb/\hbar} - e^{-2ipb/\hbar} \right]}_{\sin(2pb/\hbar)}$$

$$|\phi(p)|^2 = \frac{2A^2\hbar}{p\pi} \sin^2\left(\frac{2pb}{\hbar}\right)$$

3. **Heisenberg uncertainty** (25 points)

(a) (10 points)

$$
\begin{aligned}
p &= \frac{h}{R} && (9)\\
E^2 &= m^2c^4 + p^2c^2 = m^2c^4 + \frac{h^2c^2}{R^2} && (10)\\
\frac{hc}{R} &= \frac{6.6\times10^{-34}\ \text{J·s}\cdot 3\times10^8\ \text{m/s}}{10^{-15}\ \text{m}} = 2.0\times10^{-10}\ \text{J} = 1.2\ \text{GeV} && (11)\\
& && (12)
\end{aligned}
$$

The rest mass of neutron,

$$m_nc^2 = m_pc^2 = 0.9\ \text{GeV}\ . \qquad (13)$$

(b) (15 points) Non-relativistic calculation because $E = 1$ keV $\ll mc^2$.
Length of electron pulse $x = vt$.
From Heisenberg uncertainty $\Delta p = \dfrac{\hbar}{2x}$.

$$\frac{\Delta v}{v} = \frac{\Delta p}{p} = \frac{\hbar}{2x}\frac{1}{mv} = \frac{\hbar}{2mv^2t} = \frac{\hbar}{4Et} = \frac{1.05\times10^{-34}\ \text{J·s}}{4\times10^3\cdot1.6\times10^{-19}\ \text{J}\cdot10{-}9\ \text{s}} = 1.6\times10^{-10} \qquad (14)$$

4. **Double slit experiment** (15 points)

(a) (5 points)

$$
\begin{aligned}
\Psi_1 &= Ae^{i\varphi_1} && (15)\\
\Psi_2 &= Ae^{i\varphi_2} && (16)\\
|\Psi_D|^2 &= 2|A|^2\left(1+\cos\left(k(L_1-L_2)+\phi_1-\phi_2\right)\right) && (17)
\end{aligned}
$$

Interference pattern shifts by $\Delta x = s\dfrac{\phi_1-\phi_2}{2\pi}$, contrast unchanged.

(b) (5 points)

$$
\begin{aligned}
\Psi_1 &= A && (18)\\
\Psi_2 &= \frac{1}{2}A && (19)\\
\Psi_D &= Ae^{ikL_1} + \frac{A}{2}e^{ikL_2} && (20)\\
|\Psi_D|^2 &= |A|^2\left(e^{ikL_1}+\frac{1}{2}e^{ikL_2}\right)\left(e^{-ikL_1+\frac{1}{2}e^{-ikL_2}}\right) && (21)\\
&= |A|^2\left(1+\frac{1}{2}\left(e^{ik(L_1-L_2)}+e^{-ik(L_1-L_2)}\right)+\frac{1}{4}\right) && (22)
\end{aligned}
$$

$$
\begin{aligned}
&= |A|^2 \left(\frac{5}{4} + \cos\left(k(L_1 - L_2)\right) \right) && (23)\\
&= \frac{5}{4}|A|^2 \left(1 + \frac{4}{5} \cos\left(k(L_1 - L_2)\right) \right) && (24)\\
\Longrightarrow I_{min} &= \frac{1}{5} && (25)\\
I_{max} &= \frac{9}{5} && (26)\\
\Longrightarrow C &= \frac{9/5 - 1/5}{9/5 + 1/5} = \frac{8/5}{10/5} = \frac{4}{5} && (27)
\end{aligned}
$$

Position unchanged.

(c) (5 points)

$$k = \frac{2\pi}{\lambda_{dB}} = \frac{2\pi p}{h} \tag{28}$$

where p is electron/muon momentum. For constant energy, $p \propto \sqrt{m}$, $\lambda_{dB} \propto \frac{1}{p} \propto \frac{1}{\sqrt{m}}$

$$\Longrightarrow s \longrightarrow \frac{s}{\sqrt{207}} = \frac{s}{14.4} \tag{29}$$

Spatial period of interference pattern is reduced by factor 14.