Advanced Stochastic Processes.

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LECTURE 12

Introduction to Ito calculus

Lecture outline

- Simple processes. Ito isometry
- First 3 steps in constructing Ito integral for general processes

12.1. Ito integral for simple processes. Ito isometry

Our immediate goal is to give meaning to expressions of the form $\int X(t)dB(t) = \int X(t,\omega)dB(t,\omega)$, where $X(t)$ is some stochastic process which is adapted to the same filtration as B. As in the case of usual integration, the idea is to define $\int X(t,\omega)dB(t,\omega)$ as some kind of a limit of (random) sums $\sum_j X(t_j,\omega)(B(t_{j+1},\omega) - B(t_j,\omega))$ and show that the limit exists.

First we address the problem of $X(t)$ being adapted to the same filtration as $B(t)$. We would like to consider processes which are not necessarily continuous. Hence the Borel σ-field $\mathcal{B}$ would not work, since it was defined on the space of continuous functions. Thus, we consider a more general setting which for now we assume exists.

Setting. We have a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a filtration $\mathcal{F}_t \subset \mathcal{F}, t \in \mathbb{R}_+$. A Brownian motion process $B(t)$ is adapted to this filtration. In addition, we have another stochastic process $X(t,\omega), t \geq 0, \omega \in \Omega$ which is also adapted to the same filtration. It is assumed that for every ω the function $X(\omega) : [0,\infty) \to \mathbb{R}$ is measurable with respect to Borel σ-field on $[0,\infty)$ (but it is not necessarily continuous). Alternatively, we think of X as a mapping $X : \Omega \times [0,\infty) \to \mathbb{R}$ which is $\mathcal{F} \times \mathcal{B}$ measurable, where $\mathcal{B}$ is Borel σ-field on $[0,\infty)$.

An intuitive way to think about this setting is that for each sample $\omega \in \Omega$ we have a sample of our process $X(\cdot,\omega)$ *and* a sample of a Brownian motion $B(\cdot,\omega)$. For example we may define $X(t) = B(t), t \leq 1$ and $X(t) = 0, t > 1$. Then for each sample ω of a Brownian motion we have a sample $X(\omega)$. Note that the path of $X(\omega)$ up to time t is completely determined by the path of $B(\omega)$ up to t. But since these samples are discontinuous, we need to "lift" the probability space to non-continuous processes.

In the space of processes $X(t)$ adapted to $\mathcal{F}_t$ consider a subspace H^2 of square integrable processes. That is H^2 is the space of processes which satisfy the following property: for every $T > 0$

$$\mathbb{E}[\int_0^T X^2(t,\omega)dt] < \infty,$$

where integration is in the Riemann sense. In particular, we assume that the integral exists. Our goal is to define the Ito integral

$$I_T(X(\omega)) = \int_0^T X(t,\omega)dB(t,\omega).$$

Specifically, $I_T(X(\omega))$ is a random variable which is defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Later we will see that when we vary T we obtain that $I_T(X)$ is a stochastic process which for each T is measurable w.r.t. $\mathcal{F}_T$, i.e. it is adopted to the filtration $\mathcal{F}_t$.

Definition 12.1. A process $X \in H^2$ is called *simple* if there exists a partition $\Pi_n : 0 = t_0 < \cdots < t_n = T$ such that $X(t,\omega) = X(t_j,\omega)$ for all $t \in [t_j, t_{j+1}), 0 \le j \le n$ for all ω. The partition Π_n does not depend on ω.

The subspace of simple processes is denoted by $S^2 \subset H^2$.

As an example, fix any partition Π_n and a process $X(t)$ and consider the process $\hat{X}(t,\omega)$ defined by $\hat{X}(t,\omega) = X(t_j,\omega)$, where t_j is defined by $t \in [t_j, t_{j+1})$. In the definition it is important that $\hat{X}(t) = X(t_j)$ and not $X(t_{j+1})$. The reason is that $\hat{X}(t) = X(t_j) \in \mathcal{F}_{t_j} \subset \mathcal{F}_t$. Thus the simple process is adapted to the filtration $\mathcal{F}_t$. But $X(t_{j+1})$, while it belongs to $\mathcal{F}_{t_{j+1}}$, does not necessarily belong to $\mathcal{F}_t$, so it is not necessarily adapted.

Given a simple process $X \in S^2$ define its integral by

$$I_T(X(\omega)) = \sum_{0 \le j \le n-1} X(t_j,\omega)(B(t_{j+1},\omega) - B(t_j,\omega))$$

Theorem 12.2 (Ito isometry). *For every simple process $X \in S^2$,*

$$\mathbb{E}[I_T(X(\omega))] = 0.$$

and

$$\mathbb{E}[I_T^2(X(\omega))] = \mathbb{E}[\int_0^T X^2(t,\omega)dt].$$

In particular, $\mathbb{E}[I_T^2(X(\omega))] < \infty$, since $X \in H^2$.

For notational convenience we drop ω from now on.

Proof.

$$\begin{aligned}\mathbb{E}[I_T(X)] &= \sum_{0 \le j \le n-1} \mathbb{E}[X(t_j)B(t_{j+1}) - B(t_j)] \\ &= \sum_{0 \le j \le n-1} \mathbb{E}[X(t_j)\mathbb{E}[B(t_{j+1}) - B(t_j)|\mathcal{F}_{t_j}]] \\ &= 0,\end{aligned}$$

where in the second equality we use the tower property and the fact $X_{t_j} \in \mathcal{F}_{t_j}$. Now for the second moment we have

$$\mathbb{E}[I_T^2(X)] = \sum_{0 \le j_1, j_2 \le n-1} \mathbb{E}[X(t_{j_1})X(t_{j_2})(B(t_{j_1+1}) - B(t_{j_1}))(B(t_{j_2+1}) - B(t_{j_2}))].$$

When $j_1 < j_2$ we have

$$\mathbb{E}[X(t_{j_1})X(t_{j_2})(B(t_{j_1+1}) - B(t_{j_1}))(B(t_{j_2+1}) - B(t_{j_2}))] = 0$$

which we obtain by conditioning on $\mathcal{F}_{t_{j_2}}$, using the tower property and observing that all of the random variables involved except for $B(t_{j_2+1})$ are measurable with respect to $\mathcal{F}_{t_{j_2}}$ (recall that $\mathcal{F}_{t_{j_1}} \subset \mathcal{F}_{t_{j_2}}$).

Now when $j_1 = j_2 = j$ we have

$$\begin{aligned}\mathbb{E}[X^2(t_j)(B(t_{j+1}) - B(t_j))^2] &= \mathbb{E}[X^2(t_j)\mathbb{E}[(B(t_{j+1}) - B(t_j))^2 | \mathcal{F}_{t_j}]] \\ &= \mathbb{E}[X^2(t_j)(t_{j+1} - t_j)].\end{aligned}$$

Combining, we obtain

$$\mathbb{E}[I_T^2(X)] = \sum_j \mathbb{E}[X^2(t_j)(t_{j+1} - t_j)] = \mathbb{E}[\sum_j X^2(t_j)(t_{j+1} - t_j)] = \mathbb{E}[\int_0^T X^2(t)dt].$$

□

12.2. Constructing Ito integral for general square integrable processes

To continue our discussion we need some background regarding square integrable random variables. Given any probability space $(\Omega, \mathcal{F}, \mathbb{P})$, define $\mathcal{L}_2(\mathbb{P})$ to be the space of random variables $X : \Omega \to \mathbb{R}$ which are square integrable: $\mathbb{E}[X^2] < \infty$. One technical assumption required is that the probability space is **complete**. Namely, if $A \in \mathcal{F}$ is such that $\mathbb{P}(A) = 0$, then for every $B \subset A$; $B \in \mathcal{F}$, and, as a result, $\mathbb{P}(B) = 0$.

It turns out that this condition is w.l.g. Namely, for every probability space we can simply add all subsets of zero probability sets into $\mathcal{F}$. It is an easy exercise to show that the resulting space is again a probability space.

A sequence of random variables X_n is defined to be Cauchy if for every $\epsilon > 0$ there exists n_0 such that for all $n, m > n_0$ we have

$$\mathbb{E}[(X_n - X_m)^2] < \epsilon.$$

Theorem 12.3 (**Completeness of $\mathcal{L}_2(\mathbb{P})$**). *The space $\mathcal{L}_2(\mathbb{P})$ is* complete. *Namely, for every Cauchy sequence X_n there exists a unique random variable $X \in \mathcal{L}_2(\mathbb{P})$ such that* $\lim_{n\to\infty} \mathbb{E}[(X_n - X)^2] = 0$.

Now we use this to define $I_T(X)$. We do this in 3 steps. First we define it for the case when $X \in H^2$ is an a.s. bounded and continuous process. Then we construct it for a bounded process. Finally we approximate any process $X \in H^2$ by bounded processes and extend the definition of $I_T(X)$.

Step 1.

Proposition 1. Suppose $X \in H^2$ is a bounded continuous process: $|X| \le M$. There exists a sequence of simple processes $X_n \in S^2$ such that

$$\lim_n \mathbb{E}[\int_0^T (X_n(t) - X(t))^2 dt] = 0. \tag{12.4}$$

Proof. Fix a sequence of partitions Π_n such that $\Delta_n = \max(t_{j+1} - t_j) \to 0$ as $n \to \infty$. Given process X, consider the modified process $X_n(t) = X(t_j)$ for all $t \in [t_j, t_{j+1})$. This process is simple and is adapted to $\mathcal{F}_t$. It is also bounded by M. Therefore $X_n \in S^2$. By continuity of X, and since $\Delta_n \to 0$, we have for each realization ω and each t, $X_n(t, \omega) \to X(t, \omega)$.

Lemma 12.5. *The following holds almost surely with respect to $\omega \in \Omega$:*

$$\lim_n \int_0^T (X_n(t) - X(t))^2 dt = 0$$

Proof. This is really a result from real analysis, but we can give a probabilistic proof of it. Consider a uniform random variable $U \in [0, T]$. Then for each sample ω note that

$$\int_0^T (X_n(t) - X(t))^2 dt = T\mathbb{E}_U[(X_n(U, \omega) - X(U, \omega))^2],$$

where the expectation is only with respect to the uniform random variable U and is defined sample wise for ω. Now a.s. with respect to U we have $|X_n(U, \omega) - X(U, \omega)| \to 0$. Since $|X|, |X_n| \le M$, then applying the Bounded Convergence Theorem, we obtain $\lim_n \mathbb{E}[(X_n(U, \omega) - X(U, \omega))^2] = 0$. □

Now we define $Z_n = \int_0^T (X_n(t) - X(t))^2 dt$. We just showed $Z_n \to 0$ a.s. On the other hand $Z_n \le 4M^2T$ a.s. Again invoking the Bounded Convergence Theorem, we obtain $\lim_n \mathbb{E}[Z_n] = 0$. This completes the proof of the proposition. □

We can now define $I_T(X)$ for every bounded continuous process X as a limit of $I_T(X_n)$, where $X_n \in S^2$ is a sequence of simple processes satisfying (12.4). In order to finalize this definition, we need to show that the limit of $I_T(X_n)$ exists in some appropriate sense, and is independent from the choice of X_n (as there may be many such simple processes). Before we conduct this analysis, we extend the scope of processes X for which we define the limit. First we do this for a bounded, but not necessarily continuous process, and then for any process $X \in H^2$.

Step 2.

Proposition 2. Suppose $X \in H^2$ is a bounded process: $|X| \le M$. There exists a sequence of processes $X_n \in H^2$ such that X_n is a.s. a bounded continuous process and

$$\lim_n \mathbb{E}[\int_0^T (X_n(t) - X(t))^2 dt] = 0. \tag{12.6}$$

Proof. We use a certain "regularization" trick to turn a bounded process into a bounded continuous approximation.

Given any $n > 0$, consider any non-negative continuous function $\psi_n(t)$ such that $\psi_n(t) = 0$ for $t \in (-\infty, -\frac{1}{n}) \cup [0, \infty)$ and

$$\int_{-\frac{1}{n}}^0 \psi_n(t) dt = 1.$$

Define

$$G_n(t,\omega) = \int_{t-\frac{1}{n}}^{t} \psi_n(s-t)X(s,\omega)ds.$$

The value $G_n(t,\omega)$ is defined via values $X(s)$ with $s \le t$ and some other deterministic function ψ_n. Therefore $G_n(t) \in \mathcal{F}_t$. We need to establish that G_n is bounded continuous and that it approximates X.

Problem 1. Prove that a.s. $G_n(t,\omega)$ is a bounded continuous function. Also prove that a.s.

$$\lim_{n\to\infty} \int_0^T (X(t,\omega) - G_n(t,\omega))^2 ds = 0$$

This concludes the proof of the proposition. □

Step 3.

Proposition 3. Suppose $X \in H^2$. There exists a sequence of processes $X_n \in H^2$ such that X_n is a.s. a bounded process and

$$\lim_n \mathbb{E}[\int_0^T (X_n(t) - X(t))^2 dt] = 0. \tag{12.7}$$

Proof. Define X_n by $X_n(t) = X(t)$ when $-n \le X(t) \le n$, $X_n(t) = -n$, when $X(t) < -n$ and $X_n(t) = n$, when $X(t) > n$. Clearly $X_n(t) \in \mathcal{F}_t$. It is also bounded. We now assert (12.7). From the fact $X \in H^2$ it follows

$$\infty > c \triangleq \mathbb{E}[\int_0^T X^2(t)dt] \ge \sum_{m\ge 0} m^2 \mathbb{E}[\int_0^T 1\{m \le |X(t)| < m+1\}dt],$$

Introducing a shorthand notation

$$p_m = \mathbb{E}[\int_0^T 1\{m \le |X(t)| < m+1\}dt]$$

we have $\sum_{m\ge 0} m^2 p_m \le c < \infty$. On the other hand

$$\begin{aligned}\mathbb{E}[\int_0^T (X_n(t) - X(t))^2 dt] &= \mathbb{E}[\int_0^T (X(t) - n)^2 1\{X(t) > n\}dt] \\ &+ \mathbb{E}[\int_0^T (X(t) + n)^2 1\{X(t) < n\}dt]\end{aligned}$$

Note

$$\begin{aligned}\mathbb{E}[\int_0^T (X(t) - n)^2 1\{X(t) > n\}dt] &= \sum_{m\ge n} \mathbb{E}[\int_0^T (X(t) - n)^2 1\{m \le X(t) < m+1\}dt] \\ &\le \sum_{m\ge n} (m+1-n)^2 p_m \\ &\le \sum_{m\ge n} m^2 p_m.\end{aligned}$$

Fix $\epsilon > 0$. Since $\sum_{m\geq 0} m^2 p_m \leq c$ is finite, then there exists n_1 such that the partial sum $\sum_{m\geq n} m^2 p_m < \epsilon/2$ for all $n > n_1$. Then for all $n > n_1$

$$\mathbb{E}[\int_0^T (X(t)-n)^2 1\{X(t) > n\} dt] < \frac{\epsilon}{2}.$$

Similarly, we show that for some n_2

$$\mathbb{E}[\int_0^T (X(t)+n)^2 1\{X(t) < -n\} dt] < \frac{\epsilon}{2}$$

for all $n > n_2$. Combining,

$$\mathbb{E}[\int_0^T (X_n(t) - X(t))^2 dt] < \epsilon$$

for all $n > \max(n_1, n_2)$ and the proposition is proven. □

12.3. Additional reading materials

- Course Packet. Chapter from Harrison's book "Brownian models and stochastic control".
- Øksendal [**1**], Chapter III.

BIBLIOGRAPHY

1. B. Øksendal, *Stochastic differential equations*, Springer, 1991.