Advanced Stochastic Processes.

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LECTURE 1

Probability basics: probability space, σ-algebras, probability measure, and other scary stuff ...

Outline of Lecture

- General remarks on probability theory and stochastic processes
- Sample space Ω.
- Discrete sample space and discrete probability space.
- Continuous sample space. σ-algebra and probability measure.

1.1. General Remarks

Probability theory is devoted to studying random observations which typically take some values in $\Re$, or vectors in $\Re^d$, or integers $\mathbb{Z}$, but typically these observations are *values*.

Examples: the number of college graduates next year; average temperature on September 8; stock price of IBM tomorrow at 11am. These observation take values (have a sample space) $\mathbb{N}, \Re$ and $\Re_+ = [0, \infty)$, respectively.

The theory of stochastic processes is devoted to studying random (stochastic) observations which are *processes* described mathematically as functions.

Examples: the number of college graduates as a function of the year between 2005 and 2010; temperature on September 8 as a function of the time of the day; IBM stock price as a function of the day of the year. These random observations are *functions*: $f : [2005, 2010] \rightarrow \mathbb{N}, f : [0, 24] \rightarrow \Re, f : [0, 365] \rightarrow \Re_+$, respectively.

The focus of this class is *random (stochastic) processes.* The random processes naturally lead as well to random observations when focusing on some features of the process. These random observations (random variables) are described mathematically as *functionals* of random processes.

Examples:

- the maximum number of college graduates between 2005 and 2010: $\max_{2005\leq n\leq 2010} f(n)$;
- first time when temperature exceeded 70: $\inf\{t\in[0,24] : f(t)\geq 70\}$;
- average IBM stock price over the year: $\frac{1}{365}\int_{0\leq x\leq 365} f(x)dx$.

In order to discuss random values and random processes we need to build an abstract formulation of a *probability space* which would allow us to speak both about random values and processes.

1.2. Sample Space

Remark. The materials of this and the following lecture overlap significantly with the course Fundamentals of Probability 6.975.

The probability space is defined as a triplet $(\Omega,\mathcal{F},\mathbb{P})$ consisting of the *sample space* Ω, *σ-algebra* $\mathcal{F}$ and *probability measure* $\mathbb{P}$. We discuss these concepts in detail in the following sections.

Sample space Ω is the set of elementary events describing "what can happen". Its elements are denoted by $\omega\in\Omega$ and are defined to be *elementary outcomes.* The sample space can be finite $\Omega=\{\omega_1,\ldots,\omega_N\}$, countable $\Omega=\{\omega_1,\omega_2,\ldots,\omega_N,\ldots\}$ or uncountable (for example $\Re$).

An important sample space for this class (when we study Brownian motion) is $C([a,b])$ – the set of all continuous functions $f:[a,b]\to\Re$.

Examples.

(a) The sample space for rolling a dice once: $\Omega=\{1,2,\ldots,6\}$,
n times: $\Omega=\{1,2,\ldots,6\}^n$,
infinitely many times: $\Omega=\{1,2,\ldots,6\}^\infty$. First two are finite, last is uncountable.
Corresponding examples of elementary outcomes:
$\omega=3,\omega=\underbrace{(1,3,3\ldots,5)}_{n},\omega=(1,1,3,1,5,\ldots,2,\ldots)$.

(b) The sample space for observing a stock value at a particular time: $\Omega=\Re_+$.

(c) The sample space for observing temperature as a function of the time of the day is $\Omega=C([0,24])$. The sample space for observing stock as a function of the year is $\Omega=C([0,365])$.

Remark. The sample space for processes needs not be continuous. For example, suppose for simplicity stock are traded every business day for 7 hours between 9*am* and 4*pm*. Then the stock price on Monday and Tuesday can be described as a set of functions $f:[0,14]\to\Re_+$ which are continuous everywhere except at $t=7$ (end of trading on Monday).

Note: there is no discussion of probabilities yet. Ω simply describes the space of possible events.

1.3. Discrete Probability Space

Probability space $(\Omega,\mathcal{F},\mathbb{P})$ is easiest to define when Ω is finite or countable.

Definition 1.1. A discrete probability space is a triplet $(\Omega, \mathcal{F}, \mathbb{P})$ such that

(a) The sample space Ω is finite or countable $\Omega = \{\omega_1, \omega_2, \ldots\}$;
(b) $\mathcal{F}$ is the set of *all* subsets of Ω.
(c) $\mathbb{P}$ is a function $\mathbb{P} : \Omega \to [0,1]$ such that $\sum_{\omega \in \Omega} \mathbb{P}[\omega] = 1$.

For every subset $A \in \mathcal{F}$ we define $\mathbb{P}[A] = \sum_{\omega \in A} \mathbb{P}[\omega]$.

There is no need to describe $\mathcal{F}$ since it is the set of all subsets. Thus, we may simply write $(\Omega, \mathbb{P})$.

Examples.

(a) n unbiased coin tosses. $\Omega = \{0,1\}^n$. $\mathbb{P}[\omega] = 1/2^n$.
Suppose

$$A = \{\omega = (\omega_1, \ldots, \omega_n) : \omega_1 = 1, \omega_n = 0\}.$$

Then

$$\mathbb{P}[A] = \sum_{\omega : \omega_1 = 1, \omega_n = 0} \mathbb{P}[\omega] = 2^{n-2} \frac{1}{2^n} = \frac{1}{4}.$$

(b) Geometric random variable. A parameter $0 < \rho < 1$ is fixed. $\Omega = \mathbb{Z}_+$. For every $\omega \in \Omega, \mathbb{P}[\omega] = (1-\rho)\rho^{\omega}$.
(c) Poisson random variable. A parameter $\lambda > 0$ is fixed. $\Omega = \mathbb{Z}_+$. For every $\omega \in \Omega, \mathbb{P}[\omega] = \frac{\lambda^{\omega}}{\omega!} e^{-\lambda}$.

1.4. σ-algebras

When Ω is not countable it is not clear how to assign probabilities to elementary events.

For example, we want to capture the intuition that a randomly chosen point from $[0,1]$ has a probability $1/4$ of falling into $[1/2, 3/4]$. If we assign $\mathbb{P}[\omega]$ to be a positive value $c > 0$ then we have a problem: the total sum $\sum_{\omega} \mathbb{P}[w] = \infty$. If we assign $\mathbb{P}[\omega] = 0$, how do we get $\mathbb{P}[1/2, 3/4] = 1/4$?

The solution is to assign probabilities to *sets* $A \subset [0,1]$ of elementary outcomes, instead of individual outcomes $\omega \in [0,1]$.

For every $A \subset \Omega$ A^c denotes the complement set: $\{\omega \in \Omega : \omega \notin A\}$. A collection of sets $A_1, A_2, \ldots \subset \Omega$ is called mutually exclusive if $A_i \cap A_j = \emptyset$ whenever $i \neq j$.

Definition 1.2. A σ-field $\mathcal{F}$ is a collection of subsets of Ω satisfying the following conditions:

(a) $\Omega \in \mathcal{F}$
(b) If $A \in \mathcal{F}$ then $A^c \in \mathcal{F}$.
(c) If a sequence $A_1, A_2, \ldots \in \mathcal{F}$ then $\cup_i A_i \in \mathcal{F}$.

Any set $A \in \mathcal{F}$ is called *measurable* or $\mathcal{F}$-measurable.

σ-field replaces the set of all subsets $A \subset \Omega$ when Ω is no longer countable.

Exercise 1. Given a σ-field $\mathcal{F}$. Prove that if $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$. In general, given an infinite sequence $A_1, A_2, \ldots, \in \mathcal{F}$ prove that $\cap_i A_i \in \mathcal{F}$.

Check that the following are indeed σ-fields.

Example :

(a) Trivial σ-field $\mathcal{F} = \{\emptyset, \Omega\}$.
(b) Set $A \subset \mathcal{F}$ is fixed. $\mathcal{F} = \{\emptyset, A, A^c, \Omega\}$.
(c) The set of all subsets of the sample space: $\mathcal{F} = \{A \subset \Omega\} = 2^{\Omega}$.
(d) $\Omega = \{1, 2, \ldots, 6\}^n$ – rolling a dice n times. Let $A = \{\omega = (\omega_1, \ldots \omega_n) : \omega_1 \leq 2\}, B = \{\omega = (\omega_1, \ldots, \omega_n) : 3 \leq \omega_1 \leq 4\}, C = \{\omega = (\omega_1, \ldots, \omega_n) : \omega_1 \geq 4\}$. Then $\mathcal{F} = \{\emptyset, A, B, C, A \cup B, A \cup C, B \cup C, \Omega\}$ is a σ-field.

Proposition 1. Let $\mathcal{F}_s, s \in S$ be some set of σ-fields defined on the same sample space Ω. (Note that the index set S needs not be finite or countable). Define $\mathcal{F} = \cap_s \mathcal{F}_s$. That is a subset $A \in \mathcal{F}$ if and only if $A \in \mathcal{F}_s$ for every σ-field $\mathcal{F}_s$. Then $\mathcal{F}$ is also a σ-field.

Proof. Note that $\mathcal{F}$ is nonempty since $\Omega \in \cap_s \mathcal{F}_s$. If $A \in \mathcal{F}$ then $A \in \mathcal{F}_s$ for every s. Since $\mathcal{F}_s$ is a σ-field, then $A^c \in \mathcal{F}_s$ for every s. Therefore $A^c \in \mathcal{F}$. Similarly we show that if $A_i \in \mathcal{F}, i = 1, 2, \ldots$ then $\cap_{i \geq 1} A_i \in \mathcal{F}$. Thus $\mathcal{F}$ is a σ-field. □

What is an appropriate σ-field corresponding to the set of all intervals $[a, b] \subset [0, 1]$? What about earlier examples of random processes: temperature during the day $f : [0, 24] \rightarrow \Re$ or stock during the year $f : [0, 365] \rightarrow \Re_+$? Describing these σ-fields explicitly is not easy, so we resort to the tool of *generated σ-field.*

Given a collection $\mathcal{B}$ of subsets of Ω (which is not necessarily a σ-field) there is always at least one σ-field which contains it: simply take the set of all subsets of Ω. It is certainly a σ-field. Denote, by $\mathcal{F}(\mathcal{B})$ the smallest σ-field containing $\mathcal{B}$. It is obtained by taking intersection $\cap_s \mathcal{F}_s$ of all σ-fields $\mathcal{F}_s$ which contain $\mathcal{B} : \mathcal{F}_s \supset \mathcal{B}$. We say that $\mathcal{F}(\mathcal{B})$ is generated by $\mathcal{B}$.

1.5. Three important σ-fields

1.5.1. Borel σ-field on $[0, 1]$

Definition 1.3. *Borel* σ-field on $\Omega = [0, 1]$ is defined to be the σ-field generated by the set of all closed intervals $[a, b] \subset [0, 1]$. It is denoted by $\mathcal{B}$.

Properties of the Borel field $\mathcal{B}$

- Every interval $[a, b] \in \mathcal{B}$ by definition.
- Every interval $[0, a)$ is the complement set of $[a, 1]$ and therefore belongs to $\mathcal{B}$. Similarly, $(a, 1] \in \mathcal{B}$.
- Every interval $(a, b) = [0, b) \cap (a, 1] \in \mathcal{B}$ (use Exercise 1).
- Every point $\{a\} = [0, a] \cap [a, 1] \in \mathcal{B}$ (again use Exercise 1).
- Every union of open or closed intervals $(a_1, b_1) \cup [a_2, b_2) \cup \cdots \cup [a_n, b_n) \in \mathcal{B}$.

Exercise 2. Let $\mathbb{Q}[0, 1]$ be the set of all rational values $r \in [0, 1]$. Prove that $\mathbb{Q}[0, 1] \in \mathcal{B}$.

Is the Borel σ-field the same as the set of all subsets of $[0, 1]$? The answer is "no". There exists $H^* \subset [0, 1]$ which does not belong to $\mathcal{B}$, but constructing this set is a very non-trivial task. A convenient rule of thumb is: any easily describable subset of $[0, 1]$ is measurable!

Borel σ-field can be similarly defined on the whole real line $\Re$ instead of $[0,1]$ as the σ-field generated by the set of all intervals $[a,b] \subset (-\infty,\infty)$; or in Euclidian space $\Re^d$, as σ-field generated by rectangles $[a_1,b_1] \times \cdots \times [a_d,b_d]$.

1.5.2. σ-field on $\{0,1\}^\infty$

Set $\Omega = \{0,1\}^\infty$. Given a sequence $\omega_1,\ldots,\omega_m \in \{0,1\}$ denote by $A_m(\omega)$ the set of all infinite sequences $\omega \in \Omega$ which agree with $\omega_1,\ldots,\omega_m$ on first m coordinates. For example say $m=4$ and $\omega_1=\omega_2=\omega_3=\omega_4=0$. Then $A_4(\omega)$ is the set of all sequences starting with four zeros.

Definition 1.4. The σ-field $\mathcal{F}_\infty$ generated by all sets $A(\omega)$ when m and $\omega_1,\ldots,\omega_m$ vary arbitrarily is defined to be the *product* σ-field on $\Omega = \{0,1\}^\infty$. The sets $A(\omega)$ are called *cylinder* sets.

Similar product type σ-fields can be defined on $\Re^\infty$ or in general on any infinite product Ω^∞ of a given sample space Ω equipped with some σ-field $\mathcal{F}$. The product σ-field $\mathcal{F}_\infty$ is the σ-field generated by *cylinder* sets which are all sets of the form $A_1 \times A_2 \times \cdots \times A_m \times \Omega^\infty$, where $A_1, A_2, \ldots, A_m$ are arbitrary sets from $\mathcal{F}$. Namely, cylinder sets are sets of infinite sequences $(\omega_1,\omega_2,\ldots,\omega_m,\ldots)$ where the first element must belong to A_1, the second element to A_2, and so on, the m-th element to A_m, but the remaining elements are unrestricted. For example when $\Omega = \mathbb{R}^\infty$ we can take as cylinder sets the sets of the form $[a_1,b_1]\times[a_2,b_2]\times\cdots\times[a_m,b_m]\times\mathbb{R}^\infty$.

The product σ-field allows us to speak about discrete time processes where index m corresponds to event occurring at time m. Describing explicitly product σ-field (that is without alluding to "generating") is not possible, but we can describe some useful special cases. In general, just as in the case of $\Omega=[0,1]$, the rule of thumb is: most of the imaginable subsets of Ω^∞ belong to $\mathcal{F}_\infty$.

Proposition 2. Let $\Omega = \{0,1\}^\infty$ and $\mathcal{F}_\infty$ be the corresponding product σ-field. The following sets belong to $\mathcal{F}_\infty$.

(a) Fix any element $\omega = (\omega_1,\omega_2,\ldots) \in \Omega$. Then $\{\omega\} \in \mathcal{F}_\infty$.
(b) Fix any m. Then $\{\omega \in \Omega : \omega_k = 0, \text{ for all } k \ge m\} \in \mathcal{F}_\infty$.
(c) Fix any value $0 \le c \le 1$. Then $A(c) = \{\omega \in \Omega : \lim_{n\to\infty} \frac{\sum_{1\le i\le n}\omega_i}{n} = c\} \in \mathcal{F}_\infty$.

Proof. (a) Given a fixed $\omega = (\omega_1,\omega_2,\ldots)$, consider the sequence of sets $A_m = \{\omega' : \omega'_1 = \omega_1,\ldots,\omega'_m = \omega\} \subset \Omega$. By definition $A_m \in \mathcal{F}_\infty$ as it is a cylinder set. Applying the result of Exercise 1, $\cap_m A_m \in \mathcal{F}_\infty$. But $\cap_m A_m$ consists of exactly one point which is $\{\omega\}$.

(b) Fix any infinite sequence $\omega = (\omega_1,\omega_2,\ldots)$ such that all of the entries $\omega_k = 0$ for $k \ge m$. By the result of part (a), $\{\omega\} \in \mathcal{F}_\infty$. But the set of interest, $\{\omega \in \Omega : \omega_k = 0, \text{ for all } k \ge m\}$ is obtained as a union of exactly 2^m such sequences. By rule (c) of Definition 1.2, this set belongs to $\mathcal{F}_\infty$.

(c) This is a more difficult, but very important part of the proposition. For every pair $r, m \in \mathbb{N}$ define

$$A_{r,m} = \{\omega \in \{0,1\}^\infty : |\frac{1}{m}\sum_{1\le i\le m}\omega_i - c| \le \frac{1}{r}\}.$$

Exercise 3. Prove that $A_{r,m} \in \mathcal{F}_\infty$.

Then for every m, $B_{r,m} \triangleq \cap_{m'\ge m} A_{r,m'} \in \mathcal{F}_\infty$. Therefore $C_r \triangleq \cup_{m\ge 1} B_{r,m} \in \mathcal{F}_\infty$. We claim that

$$A(c) = \cap_{r\ge 1} C_r.$$

Exercise 4. Prove this identity.

Therefore $A(c)$ is $\mathcal{F}_\infty$-measurable.

□

1.5.3. σ-field $C[0,\infty)$

The sample space $\Omega = C[0,\infty)$ of real valued continuous functions will be an important space when we discuss Brownian motion. In order to introduce the σ-field of interest on $C[0,\infty)$ we first introduce a notation. For every $T > 0, \rho > 0$ and for every element $x \in C[0,\infty)$ define

$$B(x,\rho,T) = \{y \in C[0,\infty) : \sup_{0\le t\le T} |x(t) - y(t)| \le \rho\}.$$

Definition 1.5. The Borel σ-field on the sample space $C[0,\infty)$ (denoted $\mathcal{B}$) is the σ-field generated by all the sets $B(x,\rho,T)$, where $x \in C[0,\infty), \rho, T \in \mathbb{R}_+$ vary arbitrarily.

We now give some examples of the sets in the Borel σ-field.

Proposition 3. The following sets belong to $\mathcal{B}$:
(a) The set of all non-negative functions $\{x : x(t) \ge 0, \text{ for all } t \ge 0\}$
(b) $\{x : \lim_{t\to\infty} x(t) = \infty\}$.
(c) $\{x : x(t) \le t, \text{ for all } t\}$.

As before, most of the imaginable susbsets of $C[0,\infty)$ belong to the Borel σ-field, but not all of them. (we do not construct a counterexample).

Proof. (a) For every positive integer n let $g_n(t) = n$ for all t. Fix any positive integer m and consider the union $B(m) \triangleq \cup_n B(g_n, n, m)$. This is the set of all functions f which take value between 0 and $2n$ for all $0 \le t \le m$ for some n (see **Figure**). We have $B(m) \in \mathcal{B}$. By one of the fundamental theorems of mathematical analysis, every continuous function defined on $[0,m]$ achieves a maximum at some point $t \in [0,m]$, and, in particular, is bounded. Thus $B(m)$ is simply the set of all continuous functions which are non-negative on $[0,m]$. Now we take $\cap_{m\ge 1} B(m)$ and note that it belongs to $\mathcal{B}$ and is the set of all non-negative functions.

(b)

Exercise 5. Prove the last two parts.

□

1.6. Probability measure

We finally get to the point of discussing probabilities and random observations. We already described how to define a probability space when Ω is countable. There we assigned probability values to individual elements $\omega \in \Omega$. As we discussed before, this does not work when Ω is uncountable. The solution is to assign probabilities to *events* $A \subset \Omega$, specifically elements A of some σ-field $\mathcal{F}$, and require that it satisfies certain "natural" properties we expect to hold for probabilities.

Definition 1.6. A probability space is a triplet $(\Omega, \mathcal{F}, \mathbb{P})$ consisting of a sample space Ω, a σ-field $\mathcal{F}$ defined on Ω and a probability function (usually called *probability measure*) $\mathbb{P} : \mathcal{F} \to [0, 1]$ satisfying the following properties:

(a) $\mathbb{P}(\Omega) = 1$.
(b) For every infinite sequence $A_1, A_2, \ldots \in \mathcal{F}$, there holds $\mathbb{P}(\cup_i A_i) \le \sum_i \mathbb{P}(A_i)$.
(c) For every mutually exclusive infinite sequence $A_1, A_2, \ldots \in \mathcal{F}$ there holds $\mathbb{P}(\cup_i A_i) = \sum_i \mathbb{P}(A_i)$. The elements $A \in \mathcal{F}$ are called the events, and $\mathbb{P}(A)$ is called the probability of the event A.

The definition intuitively means the following. The probability of "anything that is possible to happen" is $\mathbb{P}(\omega \in \Omega) = \mathbb{P}(\Omega)$ is unity. If we have several events $A_1, A_2, \ldots$ then probability that "any of them happen" is at most the sum of probabilities of individual events. The inequality is used instead of equality because of the possibility of overlap. For example the probability that the temperature today is at most 70^o or temperature at $1pm$ is 60^o is *at most* the sum of probabilities of these individual events, since both can happen. But if the events are mutually exclusive, then the equality holds. Say A is "maximum temperature is at most 70^o" and B is "smallest temperature is at least 75^o". Both events cannot happen: $A \cap B = \emptyset$. Thus $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$.

An event $A \in \mathcal{F}$ is said to happen *almost surely* if $\mathbb{P}(A) = 1$.

Here are several important implications.

Proposition 4. (a) For every two events $A, B \in \mathcal{F}$, there holds $\mathbb{P}(A \cap B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cup B)$.
(b) Suppose $A_1 \subset A_2 \subset A_3 \subset \cdots$. Then $\mathbb{P}(\cup A_i) = \lim_{i\to\infty} \mathbb{P}(A_i)$.

Proof. (a) The sets $A \setminus B, B \setminus A, A \cap B$ are mutually exclusive and their union is $A \cup B$. Therefore

$$\begin{aligned}\mathbb{P}(A \cup B) &= \mathbb{P}(A \setminus B) + \mathbb{P}(B \setminus A) + \mathbb{P}(A \cap B)\\ &= \mathbb{P}(A \setminus B) + \mathbb{P}(A \cap B) + \mathbb{P}(B \setminus A) + \mathbb{P}(A \cap B) - \mathbb{P}(A \cap B)\end{aligned}$$

But $\mathbb{P}(A \setminus B) + \mathbb{P}(A \cap B) = \mathbb{P}(A)$ and $\mathbb{P}(B \setminus A) + \mathbb{P}(A \cap B) = \mathbb{P}(B)$. Therefore $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$.

(b) Set, for convenience $A_0 = \emptyset$. The events $A_1 \setminus A_0, A_2 \setminus A_1, A_3 \setminus A_2, \ldots,$ are mutually exclusive. Therefore $\mathbb{P}(\cup_i (A_i \setminus A_{i-1})) = \sum_i \mathbb{P}(A_i \setminus A_{i-1})$. But note that $\cup_i (A_i \setminus A_{i-1}) = \cup_i A_i$. Also

$$\sum_i \mathbb{P}(A_i \setminus A_{i-1}) = \lim_{n\to\infty} \sum_{i\le n} \mathbb{P}(A_i \setminus A_{i-1}).$$

But since again $A_i \setminus A_{i-1}$ are mutually exclusive, then $\sum_{i\le n} \mathbb{P}(A_i \setminus A_{i-1}) = \mathbb{P}(\cup_{i\le n}(A_i \setminus A_{i-1}))$. Finally, we observe that $\cup_{i\le n}(A_i \setminus A_{i-1}) = A_n$. Therefore $\lim_n \mathbb{P}(A_n) = \mathbb{P}(\cup_i A_i)$. This concludes the proof.

□

This completes the description of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Every discussion of probability models has some underlying probability space. The tool of *probability distribution*, which is primarily considered in elementary probability courses, allows one to by-pass these more abstract concepts. However when we discuss stochastic processes in spaces like $\{0,1\}^\infty$ or $C[0,\infty)$ these concepts are indispensable.

1.7. Reading assignments

- Notes distributed in the class.
- Durrett [**1**]
- Grimmett and Stirzaker [**2**], Chapter 1.

BIBLIOGRAPHY

1. R. Durrett, *Probability: theory and examples*, Duxbury Press, second edition, 1996.
2. G. R. Grimmett and D. R. Stirzaker, *Probability and random processes*, Oxford Science Publications, 1985.