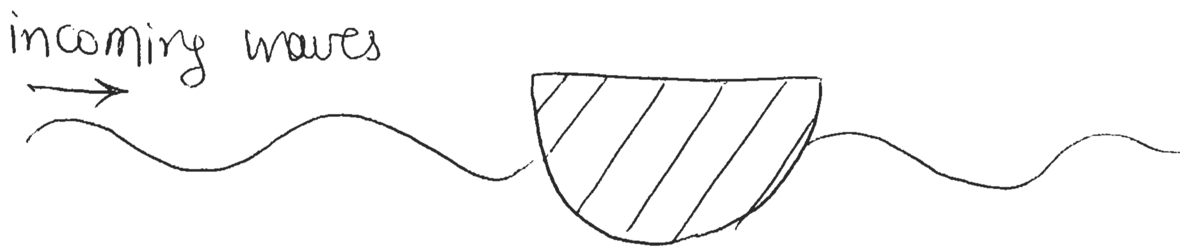
FORCES AND MOTIONS OF BODIES IN WAVES

FIRST TO IGNORE VISCOUS FORCE



Problem is complex because waves are refracted by the body.

① FIRST ASSUME SMALL WAVE SLOPE

$$a/\lambda \ll 1$$

This allows linearization of the free-surface boundary condition

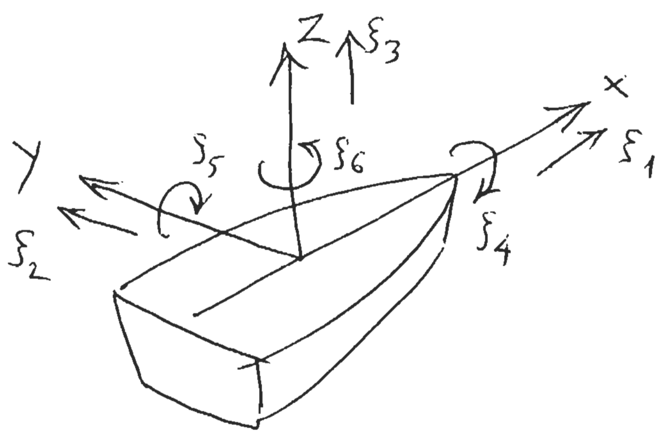
$$\boxed{\frac{\partial^2 \phi}{\partial t^2} + g\frac{\partial \phi}{\partial z} = 0 \qquad \text{at } z=0}$$

② NEXT CONSIDER SMALL MOTION FOR THE BODY

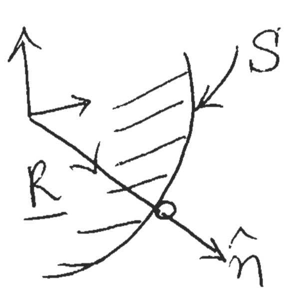
6-degree of freedom motion

$$\left.\begin{array}{ll} \text{surge} & \xi_1 \\ \text{sway} & \xi_2 \\ \text{heave} & \xi_3 \end{array}\right\} \underline{\xi}$$

$$\left.\begin{array}{ll} \text{roll} & \xi_4 \\ \text{pitch} & \xi_5 \\ \text{yaw} & \xi_6 \end{array}\right\} \underline{\Phi}$$



This allows linearization of body condition



$$V_{nf} = \frac{\partial \phi}{\partial n} = \nabla\phi \circ \hat{n} = V_{nB} = \underline{V}_B \circ \hat{n}$$

on instantaneous S

$$\boxed{\nabla\phi \cdot (\hat{n} + \underline{\Phi} \times \hat{n}) = (\underline{\xi} + \underline{\Phi} \times \underline{R}) \cdot \hat{n}}$$

THESE ASSUMPTIONS MAKE PROBLEM LINEAR

Total force consists of the sum of:

(a) Force from oncoming waves, as if the structure is not moving

(b) Force from motion ξ_j alone, as if there are no incoming waves

Further, the force from oncoming waves is split into two parts

a-(i) Force due to undisturbed waves (as if the structure is a "ghost")

a-(ii) Force due to the disturbance of the structure on the flow

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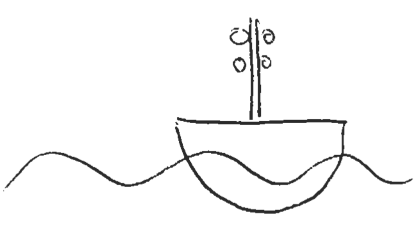
EXAMPLE: Structure heaving in waves

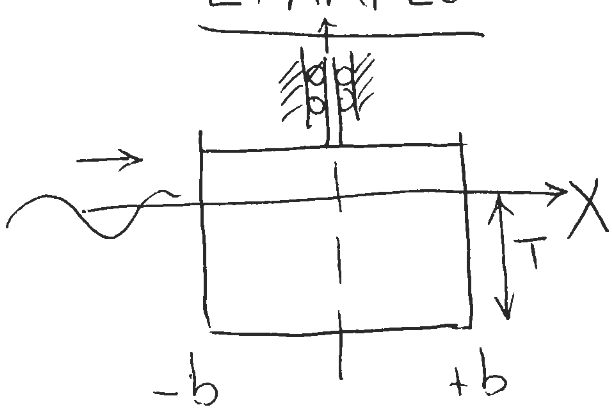


$$m\frac{d^2\xi_3}{dt^2} + \underbrace{C_{33}\xi_3}_{\text{hydrostatic}} = \underset{(a\text{-}i)}{F_{3I}} + \underset{(a\text{-}ii)}{F_{3D}} + F_{3R}$$

APROXIMATION WHEN $\lambda \gg d$

When the wavelength λ of oncoming waves is large compared to the dimension of the structure, then the force a-(i) is found by integrating the pressure P_d over the average surface $\bar{S}$, and the forces a-(ii) and (b) are approximated by the added mass force.

EXAMPLE



$$F_{3I} = \iint_{\bar{S}} P_d(x,z,t)(\hat{n}_0)_z \, dS$$

$$= -\int_{-b}^{+b} \rho \frac{\partial \phi_I(x,z,t)}{\partial t} (-1) \, dx$$

$$= \frac{a\omega^2}{K} e^{-kT} \rho \int_{-b}^{b} \cos(\omega t - kx) \, dx$$

where we used $P_d \simeq -\rho \dfrac{\partial \phi}{\partial t}$

f-5

and for ϕ the expressions for linear waves, while $(\hat{n}_0)_z = -1$ at the bottom, where $z = -T$. Note $\omega^2 = kg$ to find:

$$\Rightarrow F_{3I} = -\rho g a e^{-kT} \left[\frac{\sin(\omega t - kx)}{k} \right]_{-b}^{b}$$

$$= +2 \rho g \frac{a e^{-kT}}{k} \cos\omega t \sin kb$$

Note that if $\lambda \gg T, b$ then

$$e^{-kT} \simeq 1 - kT$$
$$\sin kb \simeq kb$$

$$\Rightarrow F_{3I} = \rho g a (1 - kT) 2b \cos\omega t$$

$$= \underbrace{\rho g (2b)}_{C_{33}} \underbrace{a \cos\omega t}_{\eta(x=0,t)} - \rho \underbrace{g k}_{\omega^2} \underbrace{(2bT)}_{\nabla} a \cos\omega t$$

$$\boxed{F_{3I} = C_{33}\, \eta(x=0,t) + \rho \nabla\, dw\, (x=0, z=-\tfrac{T}{2}, t)}$$

$$\boxed{F_{3D} \simeq + A_{33} \frac{dw}{dt}\left(x=0, z=-\frac{T}{2}, t\right)}$$

Finally

$$\boxed{F_{3R} \simeq - A_{33} \frac{d^2 \xi_3}{dt^2}}$$

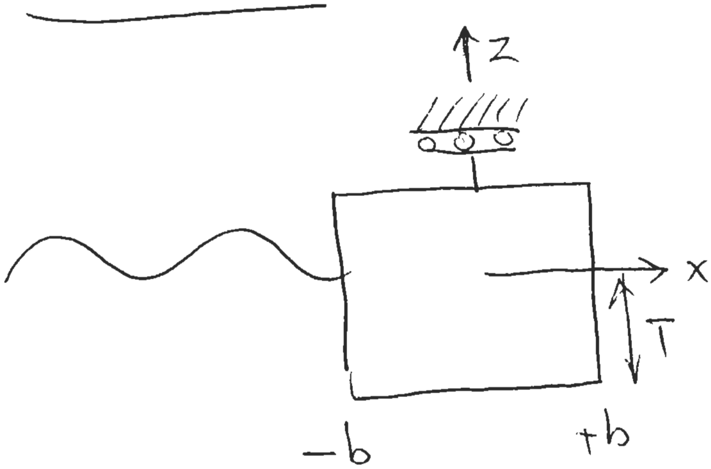
The equation of motion becomes

$$m \frac{d^2 \xi_3}{dt^2} + C_{33} \xi_3 = C_{33}\, \eta(x=0, t) + \rho\nabla \frac{dw}{dt}\left(x=0, z=-\frac{T}{2}\right.$$

$$+ A_{33} \frac{dw}{dt}\left(x=0, z=-\frac{T}{2}, t\right) - A_{33} \frac{d^2 \xi_3}{dt^2}$$

$$\boxed{\left(A_{33}+m\right)\frac{d^2 \xi_3(t)}{dt^2} + C_{33}\, \xi_3(t) = C_{33}\, \eta(x=0,t) + \left[\rho\nabla + A_{33}\right] \frac{dw}{dt}\left(x=0, z=-\frac{T}{2}, t\right)}$$

EXAMPLE



$$m\frac{d^2\xi_1}{dt^2} = F_{1I} + F_{1D} + F_{1R}$$

no hydrostatic force!

$$F_{1I} = \iint_{\bar{S}} p_d \left(\hat{n}_0\right)_1 dS = -\rho \int_{-T}^{0} \frac{\partial\phi}{\partial t}(x=-b,z,t)\,(-1)\,dz$$

$$-\rho \int_{-T}^{0} \frac{\partial\phi}{\partial t}(x=+b,z,t)\,(+1)\,dz$$

$$= \rho\frac{a\omega^2}{k}\Big\{-\cos(\omega t+kb)\int_{-T}^{0} e^{kz}\,dz$$

$$+\cos(\omega t-kb)\int_{-T}^{0} e^{kz}\,dz\Big\}$$

$$= -2\rho\frac{a\omega^2}{k}\sin\omega t\,\sin kb\,\frac{1-e^{-kT}}{k}$$

If $\lambda \gg b, T$

$$\Rightarrow \quad \sin kb \simeq kb$$

$$e^{-kT} \simeq 1 - kT \quad \Rightarrow 1 - e^{-kT} \simeq kT$$

$$F_{1I} \simeq - \rho a \omega^2 \underbrace{(2bT)}_{\nabla} \sin \omega t$$

$$\boxed{F_{1I} = \rho \nabla \frac{du}{dt}\left(x=0, z=-\frac{T}{2}, t\right)}$$

$$\boxed{F_{1D} \simeq A_{11} \frac{du}{dt}\left(x=0, z=-\frac{T}{2}, t\right)}$$

$$\boxed{F_{1R} \simeq - A_{11} \frac{d\xi_1(t)}{dt}}$$

and the equation of motion becomes

$$m \frac{d^2\xi_1}{dt^2} = \left(\rho \nabla + A_{11}\right) \frac{du}{dt}\left(x=0, z=-\frac{T}{2}, t\right) - A_{11} \frac{d\xi_1}{dt}$$

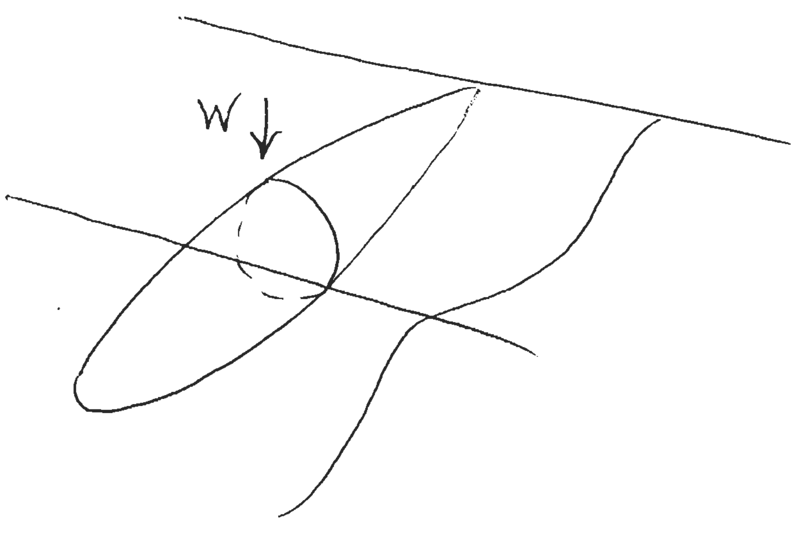
or finally

$$\boxed{(m + A_{11}) \frac{d^2 \xi_1}{dt^2} = (\rho \nabla + A_{11}) \frac{du}{dt} \left(x=0, z=-\frac{T}{2}, t \right)}$$

Note the important difference between motions with and without hydrostatic constants

VISCOSITY EFFECTS

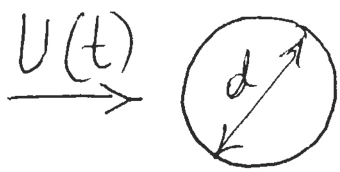
For wave forces the most important effects are caused by separation forces. For example, a streamlined vehicle when subjected to wave forces acts like a bluff body due to the transverse velocity of the waves, even in head seas.



We will use the <u>Morison formulation</u> to account for these separation forces - an empirical methodology

Morison methodology

Two-dimensional case of a cylinder in oscillatory flow.



The force in the direction of the flow consists of two components, an inertia force and a drag force

$$F(t) = (A + \rho\nabla)\frac{dU}{dt} + \frac{1}{2}\rho C_D d\, U|U|$$

where A is the added mass coefficient in the direction of the flow, $\rho\nabla$ is the displaced volume, C_D is the drag coefficient.

When both the structure and the flow move, the expression changes

$$F(t) = (\rho\nabla + A)\frac{dU}{dt} - A\frac{dv}{dt} + \frac{1}{2}\rho C_D d\,(U-v)|U-v|$$

where v is the body velocity.

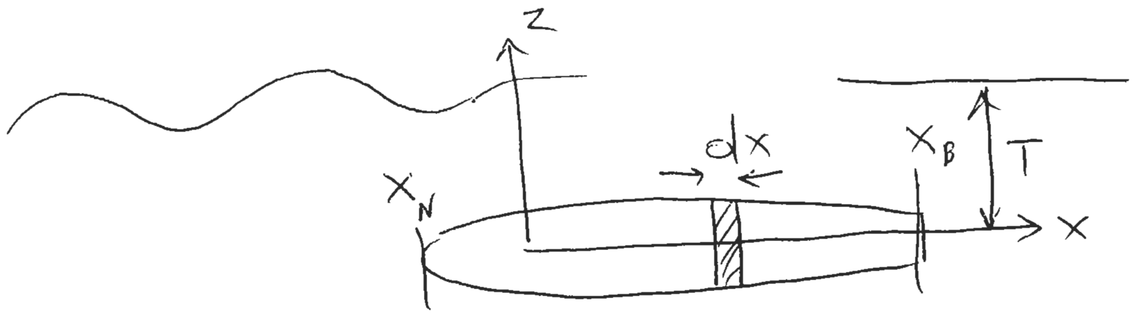
WHERE is THE TERM $\rho\nabla \frac{dU}{dt}$ coming from?

This is called the dynamic Archimedes force, because it can be derived on the same principles as the Archimedes hydrostatic force. It is very important

PROOF: In order for the fluid to accelerate there must be a pressure gradient. If the body did not exist, in its place an equal volume of water ($\rho\nabla$) would accelerate with $\frac{dU}{dt}$, hence this is the force transmitted to it by the surrounding fluid, i.e. $\rho\nabla\frac{dU}{dt}$. Hence this is the force also felt by the cylinder.

Note that this is a force that we encountered also in the previous formulation of forces in waves!

STRIP THEORY



To find the force on a long structure we use strip theory, i.e. apply the Morison formulation to a strip of length dx and then integrate

$$F_Z = \int_{x_N}^{x_B} \left\{ \left[\rho \nabla(x) + A(x)\right] \frac{dw(x, z=-T, t)}{dt} + \frac{1}{2}\rho C_D\, d(x)\, w(x, z=-T, t)\,|w| \right\} dx$$

where $\nabla(x)$ is the cross-sectional area at x, $A(x)$ the added mass per unit length (two dimensional added mass), and d the diameter at x.

Example: Cylinder of diameter d

Assuming constant cross-section, we find

$$\nabla(x) = \frac{\pi}{4} d^2 \;, \quad A(x) = \rho \frac{\pi}{4} d^2$$

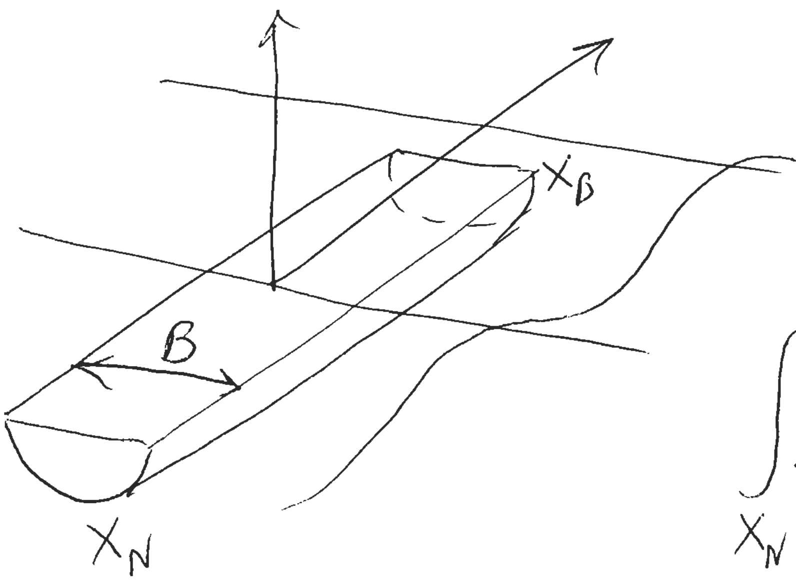
$$\Rightarrow F_2(t) = -\rho \frac{\pi}{2} d^2 \, a \omega^2 e^{-kT} \int_{x_N}^{x_B} \cos(\omega t - kx) \, dx$$

$$- \frac{1}{2} \rho d \, C_D e^{-2kT} \int_{x_N}^{x_B} \sin(\omega t - kx) \left| \sin(\omega t - kx) \right| dx$$

IMPORTANT CONSIDERATION | **FLOATING BODIES**

In floating bodies all hydrostatic terms must be accounted for and proper evaluation of the wave forces must be made. To be sure, <u>use the formulation from the previous section to</u> account for the non viscous forces

Floating cylinder in waves



The drag force remains the same, i.e.

$$\int_{x_N}^{x_B} \frac{1}{2}\rho C_D d\, w\left(x, z=-\frac{T}{2}, t\right)|w| d$$

The inertia force, however, becomes

$$\int_{x_N}^{x_B} \left\{ C_{33}\, \eta(x,t) + \left[\rho \nabla(x) + A(x)\right] \frac{dw}{dt}\left(x, z=-\frac{T}{2}, t\right) \right\}$$

where C_{33} is the two-dimensional hydrostatic force at x, $C_{33} = \rho g B(x)$ and the remaining quantities are like before $\nabla(x) = \frac{\pi}{4} d^2$, $A(x) \cong \rho \frac{\pi}{4} d^2$