Massachusetts Institute of Technology

Department of Physics

Course: 8.09 Classical Mechanics
Term: Fall 2006

Quiz 2

November 15, 2006

Instructions

- **Do not start until you are told to do so.**
- **Solve all problems.**
- Put your name on the covers of all notebooks you are using.
- Show all work neatly in the white book, label the problem you are working on.
- Mark the final answers.
- Books and notes are not to be used. You may use your calculator.

Useful Formulae

Newton and Basic Kinematics:

$$\begin{aligned}
\vec{F} &= \dot{\vec{p}} = m\vec{a} \quad \text{for} \quad v \ll c \\
\vec{v} &= \vec{v}_0 + \int_{t'=0}^{t'=t} dt' \vec{a} \\
\vec{r} &= \vec{r}_0 + \vec{v}_0 t + \int_{t'=0}^{t'=t} dt' \int_{t''=0}^{t''=t'} dt'' \vec{F}(t'')/m
\end{aligned}$$

Gravitational Law:

$$\vec{F} = -\frac{G m_1 m_2}{r_{12}^2} \hat{r}_{12}$$

Lagrangian and Hamiltonian:

$$L(q, \dot{q}) = T - U; \qquad H(p, q) = T + U = p\frac{\partial q}{\partial t} - L$$

Hamilton Equation of Motion:

$$\frac{\partial H}{\partial q} = -\dot{p}; \qquad \frac{\partial H}{\partial p} = \dot{q}$$

Generating function:

$$\frac{\partial F(Q, q)}{\partial q} = p; \qquad \frac{\partial F(Q, q)}{\partial Q} = -P$$

Poisson Brackets:

$$[g, f] = \frac{\partial g}{\partial q}\frac{\partial f}{\partial p} - \frac{\partial g}{\partial p}\frac{\partial f}{\partial q}$$

Euler-Lagrange (without and with constraints):

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0; \qquad \frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} + \lambda\frac{\partial g}{\partial x} = 0$$

Polar Coordinates:

$$x = r\sin\theta\cos\phi; \quad y = r\sin\theta\sin\phi; \quad z = r\cos\theta$$

Orbit Equation:

$$u'' + u = -\frac{\mu}{\ell^2 u^2} F(u) \quad \text{with} \quad u = \frac{1}{r}$$

Effective Potential:

$$V(r) = U(r) + \frac{\ell^2}{2\mu r^2}$$

Keplerian Orbits:

$$U(r) = -\frac{k}{r}; \qquad \frac{\alpha}{r} = \varepsilon\cos\theta + 1$$
$$\alpha = \frac{\ell^2}{\mu k}; \qquad \varepsilon = \sqrt{1 + \frac{2E\ell^2}{\mu k^2}}; \qquad \tau^2 = \frac{4\pi^2\mu}{k}a^3$$
$$r_{\min} = a(1-\varepsilon) = \frac{\alpha}{1+\varepsilon}; \qquad r_{\max} = a(1+\varepsilon) = \frac{\alpha}{1-\varepsilon}$$

Spherical Coordinates:

$$x = r\sin\theta\cos\phi; \quad y = r\sin\theta\sin\phi; \quad z = r\cos\theta$$

Inelastic Scattering (coefficient of restitution):

$$\epsilon = \frac{|v_2 - v_1|}{|u_2 - u_1|}$$

Scattering:

$$\sigma(\theta) = \frac{d\sigma}{d\Omega} = \frac{d\sigma}{d\phi\sin\theta d\theta} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$
$$\tan\psi = \frac{\sin\theta}{\cos\theta + (m_1/m_2)}; \qquad \zeta = \frac{\pi}{2} - \frac{\theta}{2};$$

Vector in fixed frame expressed in terms of vector in rotating frame and rotation velocity.

$$(\frac{d\vec{X}}{dt})_{inertial} = (\frac{d\vec{X}}{dt})_{rotating} + (\vec{\omega}\times\vec{X})$$

Acceleration in accelerated and rotating frame:

$$\vec{a} = \vec{g} - \dot{\vec{V}} - (\dot{\vec{\omega}}\times\vec{r}) - 2(\vec{\omega}\times\vec{v}) - (\vec{\omega}\times(\vec{\omega}\times\vec{r}))$$

Useful Trigonometrical Formulas

$$\sin(A+B) = \sin A\cos B + \cos A\sin B;$$
$$\cos(A+B) = \cos A\cos B - \sin A\sin B;$$

Problem 1: Canonical Transformation (35 points)

Consider a harmonic oscillator with Hamiltonian:

$$H = \frac{p^2}{2m} + \frac{m\omega^2 q^2}{2}$$

Introduce transformation of variables from $q(t)$ and $p(t)$ to $Q(t) = q(t+\tau)$ and $P(t) = p(t+\tau)$ where τ is constant. The goal of this problem is to find generating function $F(Q,q)$ of this canonical transformation of variables.

a) Solve Hamilton equations of motion and find $q(t)$ and $p(t)$ for arbitrary initial conditions.

b) Express P and p as functions of only q, Q, m, ω and τ. You may want first to express Q and P as the functions of q and p and then rearrange equations. Note that there should be no time dependence in these expressions.

c) Integrate p as a function of q to obtain q-dependence of F. Remember to include the undetermined integrating constant(q) $C(Q)$

d) Use equation for $P(Q,q)$ to find $C(Q)$ and determine $F(Q,q)$

e) Verify that $F(Q,q)$ is the correct generating function.

Problem 2: Scattering from a rotationally symmetric surface (30 points)

Consider small particles moving with velocity parallel to the z-axis. Particles are scattering from a perfectly elastic and frictionless surface. The surface is rotationally symmetric about the z-axis and the shape of the surface is $\rho(z) = z^3$ for $z > 0$. We want to find differential scattering cross section $\sigma(\theta)$.

a) Find the relationship between the scattering angle θ and the impact parameter. Note that the surface at the impact point can be considered as a perfectly elastic wall tangential to the curve at that point.

b) Extract $b(\theta)$ and determine the differential cross section.

c) Make a sketch of the scattering angle at some representative point $\rho(z)$ for the case of the surface that is frictionless but totally inelastic.

d) Make a sketch of the scattering angle at some representative point $\rho(z)$ for the case of surface that is elastic but it has very high friction.

Problem 3: Motion in rotating frame (35 points)

Consider a cannon located at geographical lattitude of exactly 45^o North. Cannon can fire identical projectiles in East, North, West and South directions. All projectiles have identical initial speed of $v_0 = 200$ m/s and the cannon always points up at exactly 45^o to the horizontal. In your calculations of the projectile trajectory you can ignore the curvature of the Earth. Assume that there is no air resistance.

Assume that the radius of the Earth is $R = 6 \cdot 10^6$ m and that the angular velocity of Earth's rotation is $\Omega = 7 \cdot 10^{-5}$ s^{-1}. Gravitational acceleration is $g = 10$ m/s^2. Do your calculations in the coordinate system where the z-axis is vertical pointing up with 0 at the center of the Earth, the x-axis points North and the y-axis points West.

a) Assume that the Earth is not rotating. Obtain the algebraic formulas and the numericals value for the total range in meters, maximum height above the ground in meters, and the projectile flight time in seconds.

b) Assume now that the Earth is rotating with its usual Ω. Write down an expression for acceleration $\vec{a}_i = (a_x, a_y, a_z)$ in terms of components of $\vec{\Omega}$, $\vec{r} = (x, y, z)$ and $\vec{v} = (\dot{x}, \dot{y}, \dot{z})$. Ignore very small terms proportional to $\Omega^2 x$, $\Omega^2 y$ and replace z with R. Keep all variables in algebraic form but replace $\sin 45^o$ and $\cos 45^o$ with $1/\sqrt{2}$.

Note: the following sections can be quite time consuming. Start them only after you have finalized the other problems and sections in this quiz

c) The gun is being shot in the four geographical directions E, N, W, S. Write down the expressions for acceleration individually for each direction. In each case remove the very small terms and find cancellations of terms of roughly equal size. Justify your choices. Note that some numerical cancellations are possible since the speed of the projectile happens to be very close to the linear speed of the Earth's surface at that lattitude.

d) Based on the magnitudes and signs of the terms of $\vec{a}$ obtained in c) estimate the effect of the earth's rotation on the position of the impact parameter of the projectile for each of the four cases. Mark the location of the new impact point as compared to no-rotation impact point (as obtained in a)) on graphs similar to Figure 1. The center of the graph corresponds to the no-rotation impact position. Do not do the precise calculations, try only to estimate the direction and relative magnitude of the deviation. All four graphs should be oriented with North at the top and East to the right.

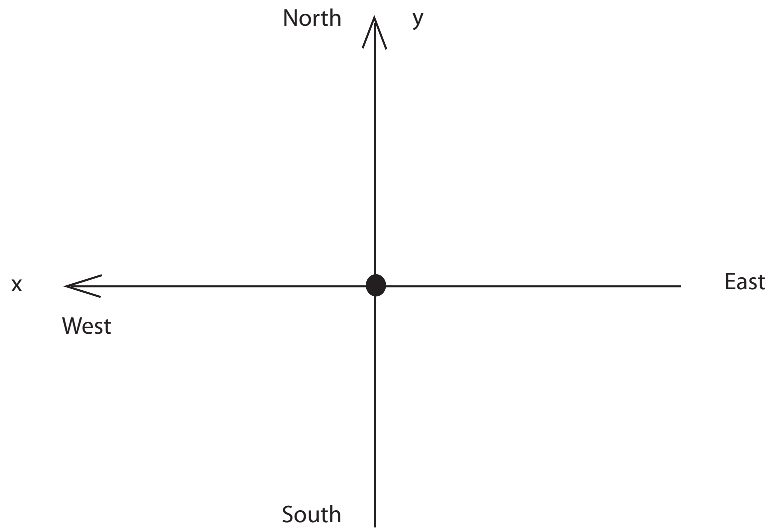

Figure 1: Impact Point Graph