# 12.003 Physics of Atmospheres and Oceans

# Problem Set 5: Equations of Motion

# Due date: Monday 29th October, 2007

1. Carefully read Chapter 6 of our notes and then answer the following questions.

2. Continuity Equation.

   In Chapter 6 of our notes the continuity equation is derived:

   $$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \,. \tag{2}$$

   (a) Using the definition of the Lagrangian derivative, show that it can be written in the alternative form:

   $$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u} = 0 \,. \tag{3}$$

   (b) The density of a compressible fluid (such as air) can change and so the flow is not, strictly speaking, non-divergent ($\nabla \cdot \mathbf{u} \neq 0$). However, show that if the flow is in hydrostatic balance, then the mass of an elemental fixed "volume" in pressure coordinates, $\delta x \times \delta y \times \delta p$, cannot change! Hence deduce that:

   $$\nabla_p \cdot \mathbf{u}_p = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0.$$

   where the subscript $p$ reminds us that we are in pressure coordinates.

3. Consider the typical, zonally averaged flow, $u$, shown here: http://paoc.mit.edu/labweb/atmos-obs/zonalwind.htm. Concentrate on the vicinity of the subtropical jet near $30^o$N in winter (DJF). If the $x$-component of the frictional force per unit mass is

   $$\mathcal{F}_x = \nu \nabla^2 u \,,$$

   where the kinematic viscosity coefficient is $\nu = 1.34 \times 10^{-5}\mathrm{m^2 s^{-1}}$ and $\nabla^2 \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$. Compare the magnitude of this eastward force with the northward or southward Coriolis force and thus convince yourself (and me!) that the frictional force is negligible. [$10^\circ$ of latitude $\simeq$ 1100km; the jet is at an altitude of about 10km. You should find that an order-of-magnitude calculation will suffice to make the point unambiguously.]

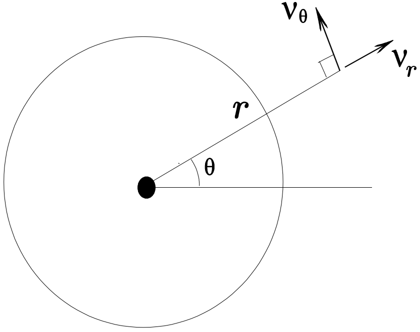
4. Using only the equation of hydrostatic balance and the rotating equation of motion, show that a fluid cannot be motionless unless its density is horizontally uniform. (Do *not* assume geostrophic balance, but you should assume that a motionless fluid is subjected to no frictional forces.)

5. Consider horizontal flow in circular geometry in a system rotating about a vertical axis with a steady angular velocity $\Omega$.

   Starting from Eq.6.29 of our notes, show that the equation of motion for the azimuthal flow in this geometry is, in the rotating frame (neglecting friction and assuming 2-dimensional flow)

$$\frac{Dv_\theta}{Dt} + 2\Omega v_r \equiv \frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\theta v_r}{r} + 2\Omega v_r = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} \tag{4}$$

   where $(v_r, v_\theta)$ are the components of velocity in the $(r, \theta)$ = (radial, azimuthal) directions (see figure). [Hint — write out Eq.6.29 in cylindrical coordinates, noting that $v_r = \frac{Dr}{Dt}$; $v_\theta = r\frac{D\theta}{Dt}$ and that the gradient operator is $\nabla = (\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial \theta})$]



   (a) Assume that the flow is axisymmetric (*i.e.*, all variables are independent of $\theta$). For such flow, angular momentum (relative to an inertial frame) is conserved. This means, since the angular momentum per unit mass is

$$m = \Omega r^2 + v_\theta r \ , \tag{5}$$

   that

$$\frac{Dm}{Dt} \equiv \frac{\partial m}{\partial t} + v_r \frac{\partial m}{\partial r} = 0 \ . \tag{6}$$

   Show that Eqs.(4) and (6) are mutually consistent for axisymmetric flow.

   (b) When water flows down the drain from a basin or a bath tub, it usually forms a vortex. It is often said that this vortex is anticlockwise in the northern hemisphere, and clockwise in the southern hemisphere. Test this saying by doing the following.

   Fill a basin or a bath tub (preferably the latter—the bigger the better) to a depth of at least 10cm, let it stand for a minute or two, and then let it drain. When a vortex forms[1], estimate, as well as you can, its angular velocity, direction, and radius (floating some small floats, such as pencil shavings, will help to see the flow). Hence calculate the angular momentum per unit mass of the vortex.

   Now, suppose that, at the instant you opened the drain, there was no motion

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[1] A clear vortex (with a "hollow" center) may not form. As long as there is an identifiable swirling motion, you will be able to proceed; if not, try repeating the experiment.

(relative to the rotating Earth). Now if only the vertical component of the Earth's rotation matters, calculate the angular momentum density due to the Earth's rotation at the perimeter of the bath tub or basin. [Your tub or basin will almost certainly not be circular, but assume it is, with an effective radius $R$ such that the area of your tub or basin is $\pi R^2$ in order to determine $m$.]

(c) Since angular momentum should be conserved, then if there was indeed no motion at the instant you pulled the plug, the maximum possible angular momentum per unit mass in the drain vortex should be the same as that at the perimeter at the initial instant (since that is where the angular momentum was greatest). Compare your answers and comment on the importance of the Earth's rotation for the drain vortex, and hence comment on the validity of the saying.