Interactive Epistemology –III

Haluk Ergin
Muhamet Yildiz

Road Map

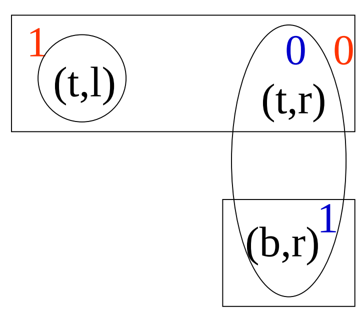
1. A state space for games
2. Epistemic Foundations for
   1. Correlated Equilibrium
   2. Rationalizability
   3. Nash Equilibrium
3. Universal Type Space

A state space for games

- Consider a family of games $\{(A,u)|u:A \rightarrow R^n\}$ for a fixed strategy space $A = A_1 \times A_2 \times \ldots \times A_n$.
- Model: $(\Omega, \mathfrak{I}_i, p_i, \mathbf{u}, s_i)_{i=1,\ldots n}$ where
  - Ω is state space
  - $\mathfrak{I}_i$ is information partition of i
  - $p_i(.|.)$ a conditional probability system for i;
  - $\mathbf{u}(\omega)$ is a profile $u:A \rightarrow R^n$ of utility functions
  - $s_i: \Omega \rightarrow A_i$
- i knows his utility function and strategy: $\omega' \in I_i(\omega) \Rightarrow s_i(\omega') = s_i(\omega)$ & $\mathbf{u}_i(\omega') = \mathbf{u}_i(\omega)$.

Example

	l	r
t	2,1	0,0
b	0,0	1,2



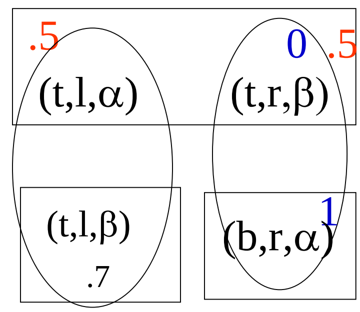
Another Example

α

	l	r
t	2,1	0,0
b	0,0	1,2

β

	l	r
t	2,2	0,3
b	3,0	1,1



(Bayesian) Rationality

A player i is said to be (Bayesian) rational at ω iff given his beliefs at ω his action at ω maximizes his expected utility, i.e., for all $a_i \in A_i$,

$$\sum_{\omega' \in I_i(\omega)} \mathbf{u}_i(\omega')\big(s_i(\omega), s_{-i}(\omega')\big) p_i(\omega' \mid I_i(\omega)) \geq \sum_{\omega' \in I_i(\omega)} \mathbf{u}_i(\omega')\big(a_i, s_{-i}(\omega')\big) p_i(\omega' \mid I_i(\omega))$$

If (A,u) is fixed, this simplifies to

$$\sum_{\omega' \in I_i(\omega)} u_i\big(s_i(\omega), s_{-i}(\omega')\big) p_i(\omega' \mid I_i(\omega)) \geq \sum_{\omega' \in I_i(\omega)} u_i\big(a_i, s_{-i}(\omega')\big) p_i(\omega' \mid I_i(\omega))$$

Common prior Assumption

$$\exists p \in \Delta(\Omega) : (\forall i)(\forall I_i),$$
$$p_i\,(.|I_i) = p(.|\,I_i)$$

Example

Underlying game

	C	D
C	(2,2)	(0,0)
D	(0,0)	(1,1)

State space

	c1	d1	d2
C1	.5,.5	.5,.5	0,0
D1	.5,.5	0,0	.5,.5
D2	0,0	.5,.5	.5,.5

Correlated Equilibrium

- Given a finite strategic form game (A,u), a ***correlated strategy profile*** consists of these elements:
  - A finite probability space (Ω',π)
  - For each player i, an information partition $\mathcal{P}_i$ of Ω'.
  - For each player i, a strategy σ_n:Ω'→A_i measurable with respect to $\mathcal{P}_i$.
- The correlated strategy profile is a ***correlated equilibrium*** if for each i and each strategy τ_i measurable with respect to $\mathcal{P}_i$,

$$\sum_{\omega\in\Omega'} u_i(\sigma(\omega))\pi(\omega) \geq \sum_{\omega\in\Omega'} u_i(\tau_i(\omega)\sigma_{-i}(\omega))\pi(\omega)$$

CPA + CK(Rationality&u) = CE

Theorem: Let $(\Omega,\mathfrak{I}_i,p_i,u,s_i)_{i=1,\ldots n}$ be a model for a game (A,u) where Ω is finite. Assume CPA: $p_1 = p_2 = \ldots = p_n = p$. If each player is rational at each ω, then the distribution of s_i is a correlated equilibrium distribution.

Proof: Define $(\Omega',\mathcal{P}_i,\pi,\sigma_i) := (\Omega,\mathfrak{I}_i,p,s_i)$. Fix i, P_i-measurable τ_i. For any ω, by rationality,

$$\sum_{\omega'\in I_i(\omega)} u_i(s_i(\omega),s_{-i}(\omega'))p(\omega'|I_i(\omega)) \geq \sum_{\omega'\in I_i(\omega)} u_i(\tau_i(w),s_{-i}(\omega'))p(\omega'|I_i(\omega)).$$

Sum over $I_i(\omega)$:

$$\sum_{\omega'\in\Omega} u_i(s_i(\omega'),s_{-i}(\omega'))p(\omega') \quad\geq\quad \sum_{\omega'\in\Omega} u_i(\tau_i(\omega'),s_{-i}(\omega'))p(\omega').$$

Sum over $I_i(\omega)$

$$\sum_{I_i(\omega)} \sum_{\omega' \in I_i(\omega)} u_i\big(s_i(\omega), s_{-i}(\omega')\big) p(\omega' | I_i(\omega))\, p(I_i(\omega)) \geq \sum_{I_i(\omega)} \sum_{\omega' \in I_i(\omega)} u_i\big(\tau_i(w), s_{-i}(\omega')\big) p(\omega' | I_i(\omega))$$

$$\updownarrow \qquad\qquad \updownarrow$$

$$\sum_{\omega' \in \Omega} u_i\big(s_i(\omega'), s_{-i}(\omega')\big) p(\omega') \quad \geq \quad \sum_{\omega' \in \Omega} u_i\big(\tau_i(\omega'), s_{-i}(\omega')\big) p(\omega')$$

CK(Rationality&u) = Rationalizability

Definition: R_i := Rationalizable strategies for i.

Theorem: Let $(\Omega, \mathfrak{I}_i, p_i, u, s_i)_{i=1,\ldots n}$ be a model for a game (A,u). If each player is rational at each ω, then $s_i(\omega) \in R_i$ for each i. Moreover, there exists a model $(\Omega, \mathfrak{I}_i, p_i, u, s_i)_{i=1,\ldots n}$ in which $s_i(\Omega) = R_i$ and each player is rational at each ω.