

14.126 Game Theory

Haluk Ergin & Muhamet Yildiz

Lecture 7:

Repeated Games

**Perfect Monitoring (continued)
Imperfect Monitoring, APS (1990)**

A Perfect Folk Theorem with Nash Threats

Theorem: (Friedman, 1971) Suppose that G has a NE a^* and let w be a payoff profile s.t. $w \gg u(a^*)$ and $w = u(\bar{a})$ for some $\bar{a} \in A$. Then there is $\bar{\delta} \in (0, 1)$ s.t. for any $\delta \in (\bar{\delta}, 1)$, w is a SPE payoff profile of $G^\delta(\infty)$.

Proof: Let s be s.t. $s(h) = \bar{a}$ if $h = \emptyset$ or $h = (\bar{a}, \bar{a}, \dots, \bar{a})$. Otherwise let $s(h) = a^*$. Then the payoff under s is $w = u(\bar{a})$. Moreover s is a SPE iff for any i and $a_i \in A$:

$$(1 - \delta)u_i(a_i, \bar{a}_{-i}) + \delta u_i(a^*) \leq u_i(\bar{a})$$

Since $u_i(a^*) < w_i = u_i(\bar{a})$, the above inequality is satisfied for all i and a_i , when δ is large enough. \square

Question: *What if w is a feasible and strictly enforceable payoff profile that is not a pure strategy NE payoff of G ?*

- Public Randomization
- Time averaging (Fudenberg and Maskin, 1991)

Strategies as Automata

An **automaton** for player i in $G(\infty)$ consists of:

- A set Q_i (the set of **states**)
- An element $q_i^1 \in Q_i$, (the **initial state**)
- A function $f_i: Q_i \rightarrow A_i$ (the **output function**)
- A function $\tau_i: Q_i \times A \rightarrow Q_i$ (the **transition function**)

The strategy s_i induced by $(Q_i, q_i^0, f_i, \tau_i)$ is given by:

- For any $h = (a^1, \dots, a^{t-1})$, find $q_i^t(h)$ inductively:
 - Let $q_i^1(\emptyset) = q_i^1$
 - For $1 < k \leq t - 1$, let

$$q_i^k(a^1, \dots, a^{k-1}) = \tau_i(q_i^{k-1}(a^1, \dots, a^{k-2}), a^{k-1})$$

- Set $s_i(h) = f_i(q_i^t(a^1, \dots, a^{t-1}))$.

Example: Tit-for-tat.

A More General Perfect Folk Theorem

Theorem: (Fudenberg and Maskin, 1986) Let \bar{a} be a strictly enforceable action profile. Assume that there is a collection $(a(i))_{i \in N}$ of strictly enforceable action profiles s.t. for every i

$$u_i(\bar{a}) > u_i(a(i)) \text{ and } u_i(a(j)) > u_i(a(i)) \text{ for } j \neq i.$$

Then there is $\bar{\delta} \in (0, 1)$ s.t. for any $\delta \in (\bar{\delta}, 1)$, there is SPE s of $G^\delta(\infty)$ in which players play \bar{a} on the equilibrium path.

Note: With public randomization our proof will also imply: *If the set of feasible payoffs has dimension n (**full dimension condition**), then any strictly enforceable feasible payoff profile is an SPE payoff profile of $G^\delta(\infty)$ for δ sufficiently close to 1.*

States: $\{C(j) \mid 0 \leq j \leq n\} \cup \{P(j, t) \mid j \in N \ \& \ 1 \leq t \leq L\}$.

Initial state: $C(0)$. (Set $a(0) := \bar{a}$.)

Output function: $a(j)$ in $C(j)$, and (p_{-j}, a_j^v) in $P(j, t)$.

Transitions: If the last play is a :

- From $C(j)$ stay in $C(j)$ unless a single player deviated from $a(j)$, in which case move to $P(j, L)$.
- From $P(j, t)$:
 - If a single player $k \neq j$ deviated from (p_{-j}, a_j^v) , then move to $P(j, L)$
 - Otherwise go to $P(j, t - 1)$ if $t > 1$; to $C(j)$ if $t = 1$.

Let M be the maximum payoff in G . Choose L, δ s.t. $\forall i, j$:

$$M - u_i(a(j)) < L[u_i(a(j)) - v_i]$$

$$M - u_i(a(j)) < \sum_{k=2}^{L+1} \delta^{k-1} [u_i(a(j)) - v_i]$$

$$\sum_{k=1}^L \delta^{k-1} [M - u_i(p_{-j}, a_j^v)] < \sum_{k=L+1}^{\infty} \delta^{k-1} [u_i(a(j)) - u_i(a(i))]$$

Other Folk Theorems for $T = \infty$

- Allowing for behavioral strategies. (F&M 1986)
- Abreu, Dutta, and Smith (1994):
 - If no two players have the same vNM preference over A (**NEU**), and if there is public randomization, then every feasible strictly enforceable profile is a SPE payoff profile in $G^\delta(\infty)$ for sufficiently large δ .
 - If there is no $\sigma = (\sigma_1, \dots, \sigma_n)$ that simultaneously gives two players their minimum payoff in G (**NSM**), then NEU is *necessary* for the perfect folk theorem.
- Alternative methods of evaluating payoffs sequences:
 - **Limit of Means** (Aumann and Shapley, 1976)
 $(v_i^1, v_i^2, \dots) \succ_i (w_i^1, w_i^2, \dots) \Leftrightarrow \liminf \frac{1}{T} \sum_{t=1}^T (v_i^t - w_i^t) > 0$
 - **Overtaking** (Rubinstein, 1979)
 $(v_i^1, v_i^2, \dots) \succ_i (w_i^1, w_i^2, \dots) \Leftrightarrow \liminf \sum_{t=1}^T (v_i^t - w_i^t) > 0$

A Perfect Folk Theorem for Finitely Repeated Games

Theorem: (Krishna, 1989) Let \bar{a} be a strictly enforceable action profile. Assume that (i) for each $i \in N$ there are two NE of G that differ in the payoff of player i and (ii) there is a collection $(a(i))_{i \in N}$ of strictly enforceable action profiles s.t. for every i

$$u_i(\bar{a}) > u_i(a(i)) \text{ and } u_i(a(j)) > u_i(a(i)) \text{ for } j \neq i.$$

Then for any $\epsilon > 0$, there is an integer T^* s.t. if $\infty > T > T^*$ then $G(T)$ has an SPE whose payoffs are within ϵ of $u(\bar{a})$.

**Repeated Games with Unobservable
Actions
(Imperfect Monitoring)**

Abreu, Pearce, Stachetti (1990)

The Stage Game G

There is a finite set of players N , each player i has finite set of actions A_i .

Publicly observable **signals** $p \in \Omega$. (Ω is finite)

$f(p|a)$: probability p conditional on the action profile a .

Realized payoffs: $\pi_i(a_i, p)$

Expected payoffs: $\Pi_i(a) = \sum_{p \in \Omega} f(p|a)\pi_i(a_i, p)$.

Player i 's expected payoffs depends on the others' actions only through the distribution of the public signal.

Important Assumptions:

- $f(\cdot|a)$ has full support for any $a \in A$.
- The stage Game G has a pure strategy NE.

The Infinitely Repeated Game

We will restrict attention to *pure strategies*. A **strategy** of player i is a sequence $(\sigma_i(1), \sigma_i(2), \dots)$ where:

$$\sigma_i(t) : \Omega^{t-1} \times A_i^{t-1} \rightarrow A_i \quad \text{for } t \geq 1.$$

$v(\sigma)$ = the **expected average discounted payoff** from the strategy profile σ .

Equilibrium payoffs:

$V = \{v(\sigma) \mid \sigma \text{ is a sequential eqm. of the repeated game}\}.$

Note:

- The one deviation property holds for the repeated game.
- V is a nonempty, and bounded subset of \mathbb{R}^N .

A Section from the Repeated Game: The Augmented Game

Given a function $u: \Omega \rightarrow \mathbb{R}^N$ (promised utilities), let

$$E(a, u) = (1 - \delta)\Pi(a) + \delta \sum_{p \in \Omega} f(p | a) u(p).$$

Defn: A pair (a, u) where $a \in A$ and $u: \Omega \rightarrow \mathbb{R}^N$ is **admissible w.r.t.** $W \subset \mathbb{R}^N$ if:

1. $u(\Omega) \subset W$, and
2. $E_i(a, u) \geq E_i(a'_i, a_{-i}; u)$ for any $i \in N$ and $a'_i \in A_i$.

Defn: $B(W) = \{E(a, u) \mid (a, u) \text{ is admissible w.r.t. } W\}$.

Defn: W is **self-generating** if $W \subset B(W)$.

Results that We Will Prove

Theorem 1 (Self-Generation) *If W is bounded and self-generating then $B(W) \subset V$. In fact $B^k(W) \subset V$.*

Note: Boundedness of W is a necessary assumption.

Theorem 2 (Factorization) $V = B(V)$. Hence V is the largest bounded fixed point of B .

Theorem 6 (Monotonicity) $\delta_2 > \delta_1 \Rightarrow V^1 \subset V^2$.

Vector Notation: $p^t = (p_1, \dots, p_t)$.

For any strategy profile σ and history $h = (p^{t-1}, a^{t-1}) \in \Omega^{t-1} \times A^{t-1}$, let $\sigma|_h$ denote the **continuation strategy profile conditional on h** , i.e.:

$$\sigma_i|_h(\tau)(\tilde{p}^{\tau-1}, \tilde{a}_i^{\tau-1}) = \sigma_i(t+\tau-1)\left(\left(p^{t-1}, \tilde{p}^{\tau-1}\right), \left(a_i^{t-1}, \tilde{a}_i^{\tau-1}\right)\right).$$

Proof of Self-Generation

For any $x \in B(W)$ let $(a(x), U(x))$ be an admissible pair w.r.t. W s.t. $x = E(a(x), U(x))$.

let $w \in B(W)$. Define the functions $U^t(w) : \Omega^t \rightarrow \mathbb{R}^N$ and the strategy profile $\sigma(w)$ recursively:

$$U^1(w)(p) = U(w)(p) \quad \text{and} \quad \sigma(w)(1) = a(w)$$

$$U^{t+1}(w)(p^{t+1}) = U \left(U^t(w)(p^t) \right) (p_{t+1})$$

$$\sigma(w)(t+1)(p^t) = a \left(U^t(w)(p^t) \right)$$

It remains to verify that:

- $v(\sigma(w)) = w$, and
- No agent has a profitable single-deviation under $\sigma(w)$.