14.126 Game Theory

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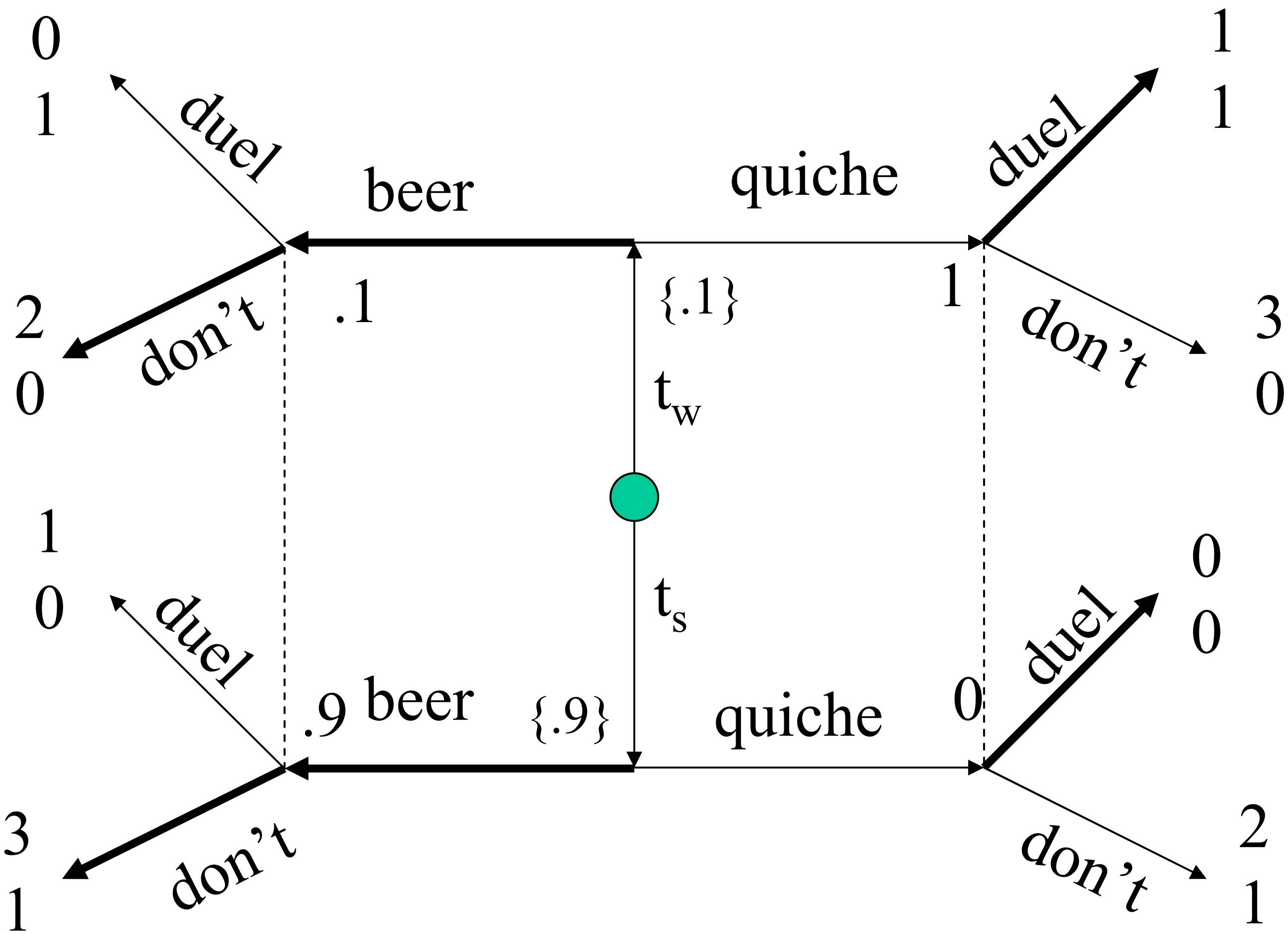
Lecture 6:

Sequential Equilibrium (continued)

Forward Induction
Repeated Games

B&Q, A Sequential Equilibrium



Single-Deviation Principle

An extensive-form game is **continuous at infinity** if, given any $\epsilon > 0$, there exists some t such that, for any two terminal histories that agree in the first t actions, the payoff difference of each player is less than ϵ.

Theorem: Let Γ be an extensive form game of perfect recall that is continuous at infinity. Suppose that $A(h)$ is finite for all $h \in H$ and let (β, μ) be a consistent assessment. Then (β, μ) is sequentially rational iff for any player i and any information set $I_i \in \mathcal{I}_i$, player i cannot increase his conditional payoff at I_i by deviating from his strategy only at I_i, i.e. for any β_i' that agrees with β_i except on I_i:

$$\mathbb{E}_{\mu(\cdot|I_i)} u_i(\mathcal{O}(\beta)|h) \geq \mathbb{E}_{\mu(\cdot|I_i)} u_i(\mathcal{O}(\beta_i', \beta_{-i})|h).$$

Sketch of "$\Leftarrow$": Suppose that (β, μ) is not sequentially rational, then there exist i, I_i^*, and β_i^* s.t.

$$\mathbb{E}_{\mu(\cdot|I_i^*)} u_i(\mathcal{O}(\beta_i^*, \beta_{-i})|h) > \mathbb{E}_{\mu(\cdot|I_i^*)} u_i(\mathcal{O}(\beta)|h) + \epsilon$$

for some $\epsilon > 0$. By continuity at infinity, there is β_i^{**} that agrees with β_i^* in finitely many info. sets, agrees with β_i in the remaining info. sets, and:

$$\mathbb{E}_{\mu(\cdot|I_i^*)} u_i(\mathcal{O}(\beta_i^{**}, \beta_{-i})|h) > \mathbb{E}_{\mu(\cdot|I_i^*)} u_i(\mathcal{O}(\beta)|h).$$

Let β_i^{***} be a profitable deviation conditional on I_i^*, which differs from β_i in the minimal number of info sets. Let I_i be s.t. β_i^{***} and β_i disagree on I_i but agree afterwards.

- Minimality implies that I_i is reached with positive probability under $(\beta_i^{***}, \beta_{-i})$ and $\mu(\cdot|I_i^*)$ conditional on I_i^*.
- Consistency implies that $\mu(\cdot|I_i)$ is derived from $(\beta_i^{***}, \beta_{-i})$ and $\mu(\cdot|I_i^*)$ conditional on I_i^* using the Bayes rule.

Hence $\beta_i^{***}(I_i)$ gives the desired profitable single deviation conditional on I_i. $\square$

Trembling Hand Perfection

A **Trembling Hand Perfect Equilibrium (THPE)** of a finite normal form game is an (ind.) mixed strategy profile σ^* s.t. there is a sequence of (ind.) mixed strategy profiles (σ^k) s.t. $\sigma^k \to \sigma^*$ and $\sigma_i^* \in B_i(\sigma_{-i}^k)$ for all i and k.

Theorem Every finite normal form game has a THPE.

In the following let Γ be any finite extensive form game with perfect recall. The **agent normal form game** of Γ is defined by $N' = \mathcal{I}$, $A_I = A(I)$, and $u_{I_i}(s) = u_i(\mathcal{O}(s))$.

Theorem For any THPE β of the agent normal form game of Γ, there is a belief system μ s.t. (β, μ) is a sequential equilibrium of Γ.

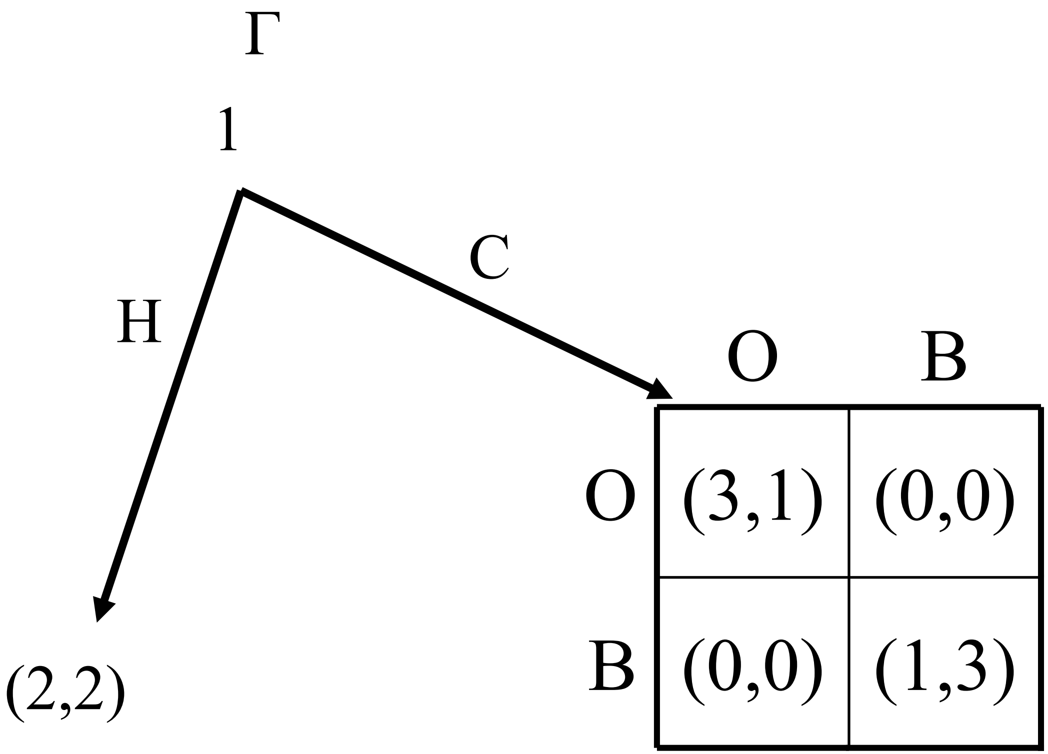
Corollary Γ has a sequential equilibrium.

Forward Induction

Strong Belief in Sequential Rationality:

If there are two pure strategies s_i and s_i' of a player i that are consistent with a history of play, and if s_i is strictly dominated but s_i' is not, at this history no player j believes that i plays s_i.

Modified BoS with Outside Option

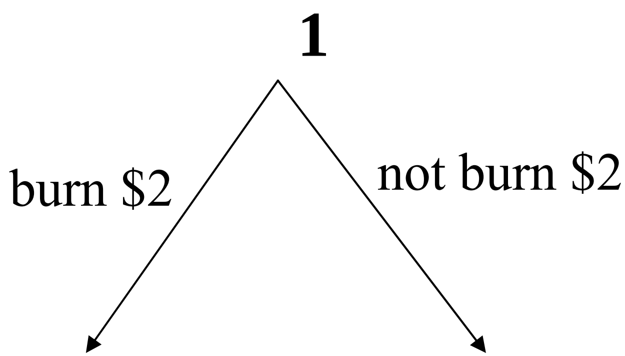


G^Γ

	O	B
HO	2,2	2,2
HB	2,2	2,2
CO	3,1	0,0
CB	0,0	1,3

Modified BoS with Money Burning

Γ



	Opera	Ballet
Opera	(2,1)	(-2,0)
Ballet	(-2,0)	(-1,4)

	Opera	Ballet
Opera	(4,1)	(0,0)
Ballet	(0,0)	(1,4)

G^{Γ}

	OO	OB	BO	BB
bOO	2,1	2,1	-2,0	-2,0
bOB	2,1	2,1	-2,0	-2,0
bBO	-2,0	-2,0	-1,4	-1,4
bBB	-2,0	-2,0	-1,4	-1,4
nOO	4,1	0,0	4,1	0,0
nOB	0,0	1,4	0,0	1,4
nBO	4,1	0,0	4,1	0,0
nBB	0,0	1,4	0,0	1,4

Repeated Games with Observable Actions
(Perfect Monitoring)

Prisonner's Dilemma

	C	D
C	3,3	0,4
D	4,0	1,1

Let $G = (N, A, u)$ be a finite normal form game. Let $G(T)$ denote the extensive form game where:

- At each stage $t = 1, 2, \ldots, T$, the game G is played among the same players in N
- Players observe the action profiles played in earlier stages,
- Payoffs in G(T) are given by:
  - If $T < \infty$ then $u_i(a_1, \ldots, a_T) = \frac{1}{T} \sum_{t=1}^{T} u_i(a^t)$.
  - If $T = \infty$ then $u_i(a_1, a_2, \ldots) = (1-\delta) \sum_{t=1}^{\infty} \delta^{t-1} u_i(a^t)$ for some common discount factor $\delta \in (0, 1)$.
- G is called the **stage game**.

Note: (1) $G(T)$ has perfect recall, it is continuous at infinity, and G is finite so the single-deviation principle applies.

(2) In the discussion of repeated games, we will *restrict attention to pure strategies* in $G(T)$.

(3) We will sometimes write $G^{\delta}(\infty)$ instead of $G(\infty)$.

Minmax, Feasible, & Enforceable Payoffs

The **minmax payoff** of player i is given by:

$$\underline{v}_i = \min_{a_{-i} \in A_{-i}} \max_{a_i \in A_i} u_i(a_i, a_{-i}).$$

Let (p_{-i}, a_i^v) be a solution of the problem above.

A payoff profile $w \in \mathbb{R}^n$ is **feasible** if it is a convex combination of payoffs $(u(a))_{a \in A}$ in G.

A payoff profile $w \in \mathbb{R}^n$ is **[strictly] enforceable** if $[w_i > \underline{v}_i]$ $w_i \geq \underline{v}_i$ for any $i \in N$. An action profile $a \in A$ is **[strictly] enforceable** if $u(a)$ is [strictly] enforceable.

Example:

	L	M	R
U	2,2	1,1	1,0
M	1,1	4,4	0,6
D	0,1	6,0	0,0

Preliminary Observations

Proposition: Let w be the payoff profile under a NE of $G(T)$. Then w is feasible and enforceable.

Proposition: Suppose that $a(t)$ is a pure strategy NE of G for ant $t = 1, \ldots, T$. Then the strategy profile s in $G(T)$ given by $s(a^1, \ldots, a^t) = a(t+1)$ for any history $(a^1, \ldots, a^t)$ and $t < T$, is an SPE of $G(T)$.

Proposition: Suppose that $T < \infty$ and G has a unique pure strategy NE a^*. Then $G(T)$ has a unique SPE strategy profile s where at each stage players play according to a^*.

Suppose that G has multiple NE and $T < \infty$. Can there be a SPE s of $G(T)$ where $s(h)$ is not a NE of the stage game for some histories h?