14.126 Game Theory

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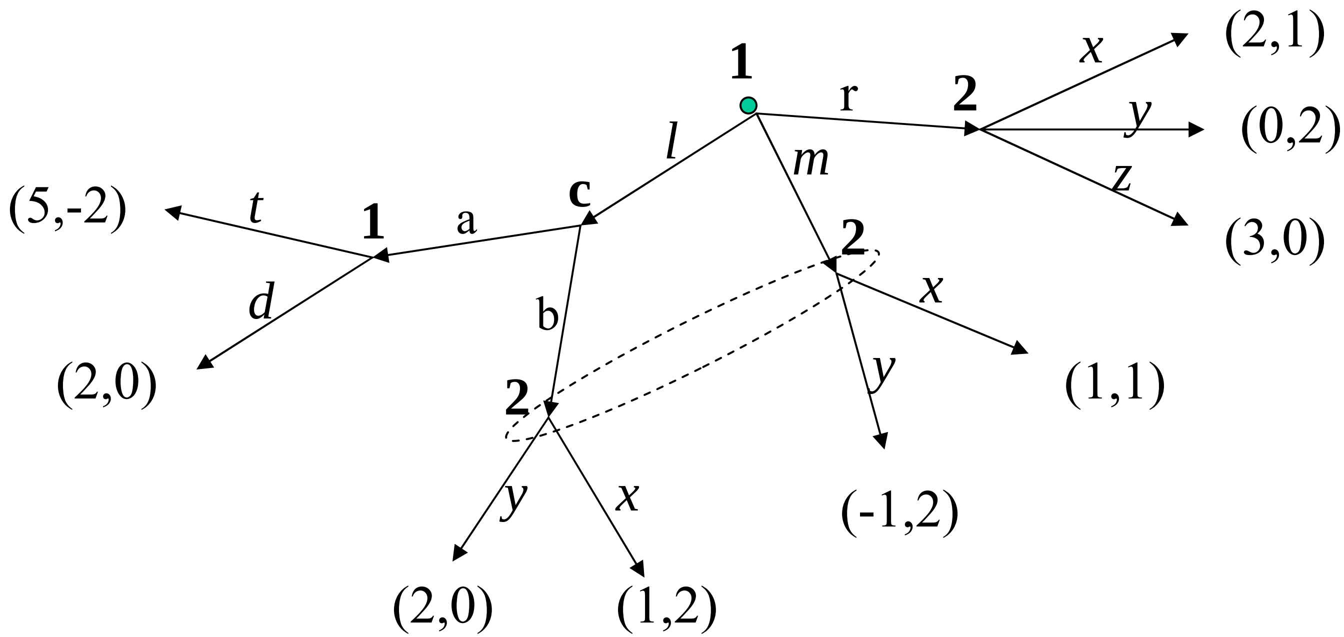
Lecture 5:

Extensive Form Games

An extensive form game



$f_c(a|(l))=f_c(b|(l))=1/2$,

$I_1=\{\{\emptyset\},\{(l,a)\}\}$, $I_2=\{\{r\},\{(m),(l,b)\}\},\ldots$

An **extensive form game** $\Gamma = (N, H, \mathcal{I}, u, P, f_c)$ consists of

- *Players:* a finite set of players $N = \{1, \ldots, n\}$,
- *Histories/Nodes:* a set of sequences H satisfying (i) the empty sequence $\emptyset$ is in H, (ii) if $(a^1, \ldots, a^k) \in H$ and $l < k$ then $(a^1, \ldots, a^l) \in H$, and (iii) if $(a^1, \ldots, a^k) \in H$ for any k, then $(a^1, a^2, \ldots) \in H$:
  - $h \in H$ is terminal if it is infinite or if there is no a s.t. $(h, a) \in H$. Z denotes terminal histories.
  - $A(h) = \{a \,|\, (h, a) \in H\}$ for each $h \in H \setminus Z$.
- *Payoffs:* a vNM payoff function $u_i \colon Z \to \mathbb{R}$ for each i,
- *Player function:* a function $P \colon H \setminus Z \to N \cup \{c\}$. If $P(h) = c$ (the chance move), a probability distribution $f_c(\cdot|h)$ over $A(h)$ is also specified.
- *Information sets:* for each i, a partition $\mathcal{I}_i$ of $\{h \in H | P(h) = i\}$ s.t. if $h, h' \in I_i \in \mathcal{I}_i$ then $A(h) = A(h')$.

(Γ is said to be **finite** if H is finite, $\mathcal{I} = \cup_{i \in N} \mathcal{I}_i$)

Perfect Recall & Perfect Information

- *An extensive form game has* **perfect information** *if all information sets are singletons:*

- *An extensive form game has* **perfect recall** *if players never forget what they knew and the actions they took earlier:*

  Formally, let $X_i(h)$ be the sequence of the information sets i visits on h and the actions he takes at them. Then perfect recall requires that $X_i(h) = X_i(h')$ whenever h and h' are in the same information set $I_i \in \mathcal{I}_i$.

Strategies

A strategy of a player is a **complete contingent-plan**, determining which action he will take at each information set he is to move, *including the information sets that will not be reached according to this strategy.*

A **pure strategy** of player $i \in N$ in an extensive form game is a function s_i that associates an action in $A(I_i)$ to each information set $I_i \in \mathcal{I}_i$. (where $A(I_i) := A(h)$ for $h \in I_i$.)

A **behavioral strategy** of player i is a collection $(\beta_i(I_i))_{I_i \in \mathcal{I}_i}$ of *independent* probability distributions, where $\beta_i(I_i)$ is a probability distribution over $A_i(I_i)$.

Nash Equilibrium

Given an extensive form game $\Gamma = (N, H, \mathcal{I}, u, P, f_c)$:

For any profile of behavioral strategies β and non-terminal history h, let

$$\mathcal{O}(\beta|h)$$

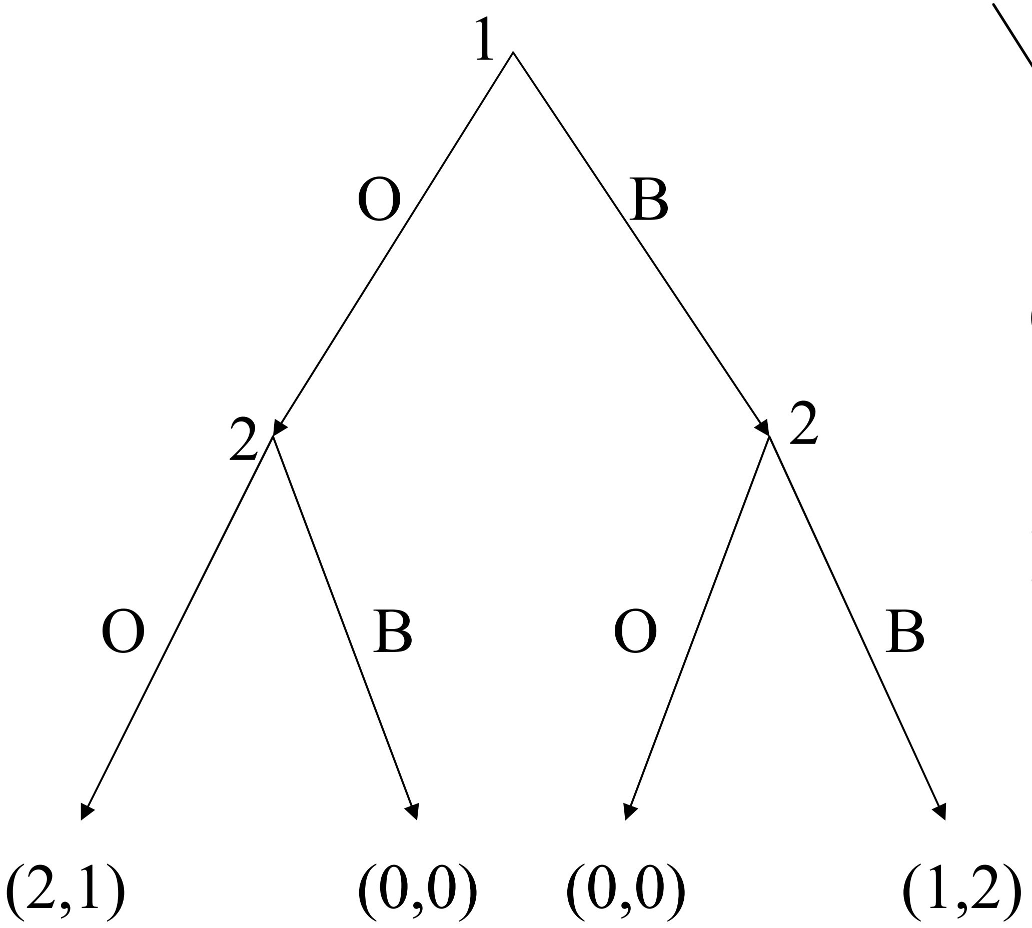
denote the distribution over Z induced by β starting at history h. ($\mathcal{O}(\beta) := \mathcal{O}(\beta|\emptyset)$, see O&R p213)

A profile of behavioral strategies β^* is a **Nash Equilibrium** if for any i and β_i:

$$u_i(\mathcal{O}(\beta^*)) \geq u_i(\mathcal{O}(\beta_i, \beta^*_{-i})).$$

Note: Let G^Γ be a normal form game where the actions of player i are his pure strategies in Γ and his payoff function is defined by $\tilde{u}_i(s) = u_i(\mathcal{O}(s))$. Then NE of Γ "correspond to" NE of G^Γ.

Battle of the Sexes with perfect information

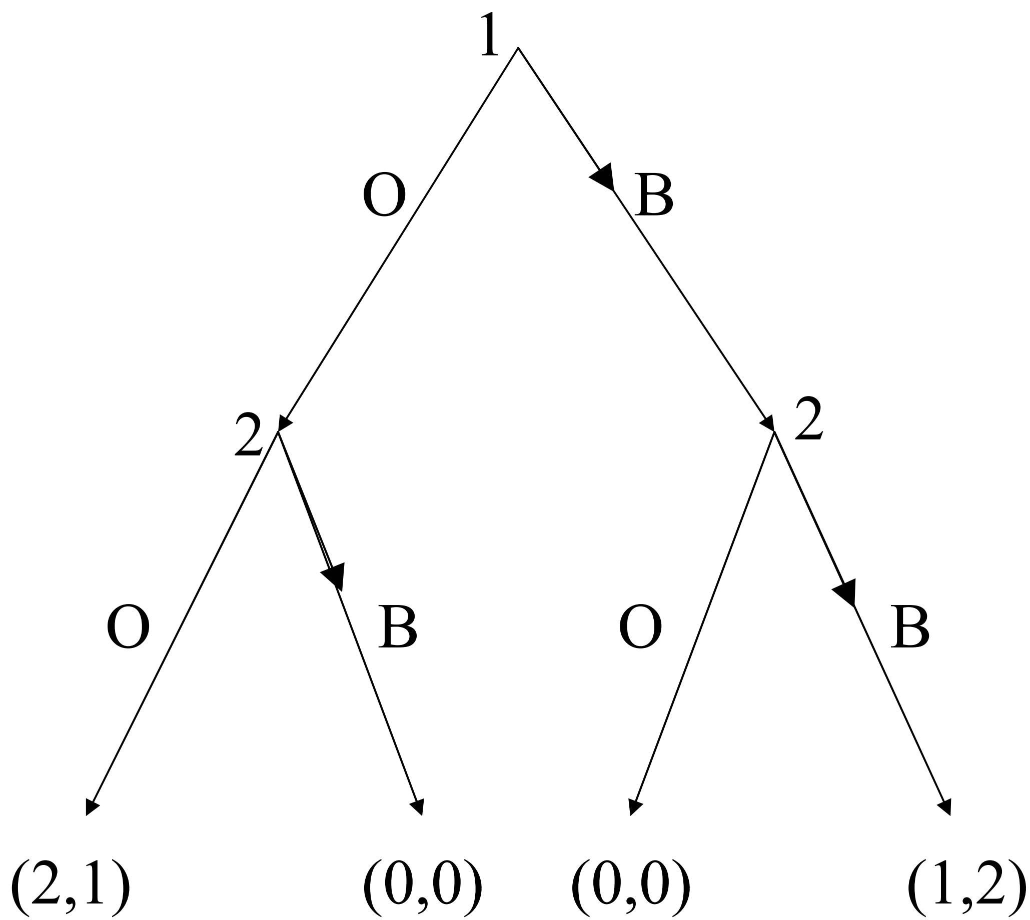


Normal Form (XY=[X|O,Y|B]) :

	OO	OB	BO	BB
O	2,1	2,1	0,0	0,0
B	0,0	1,2	0,0	1,2

Nash Equilibria?

What is wrong with this NE?



Sequential Rationality

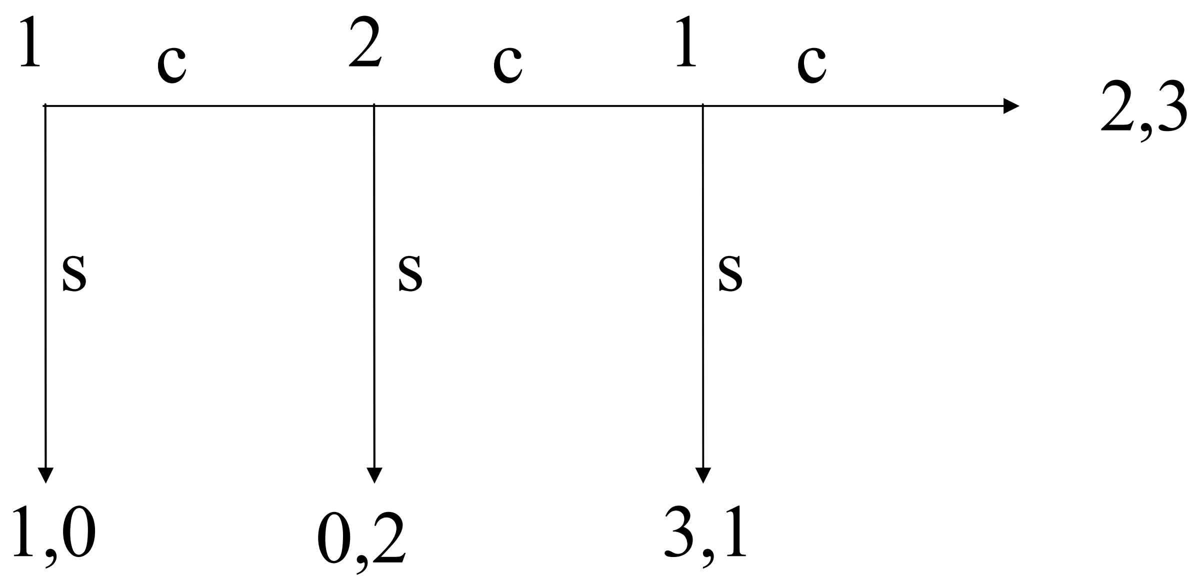
A player i is **sequentially rational** if at each of his information sets, he maximizes his continuation expected utility given his beliefs and the other players' strategies:

- for each $I_i \in \mathcal{I}_i$, there is a probability distribution $\mu(\cdot|I_i)$ over the histories in I_i representing i's beliefs,
- given the others' strategies β_{-i}, for each $I_i \in \mathcal{I}_i$, β_i maximizes $\mathbb{E}_{\mu(\cdot|I_i)} u_i(\mathcal{O}(\beta_i, \beta_{-i}|h))$.

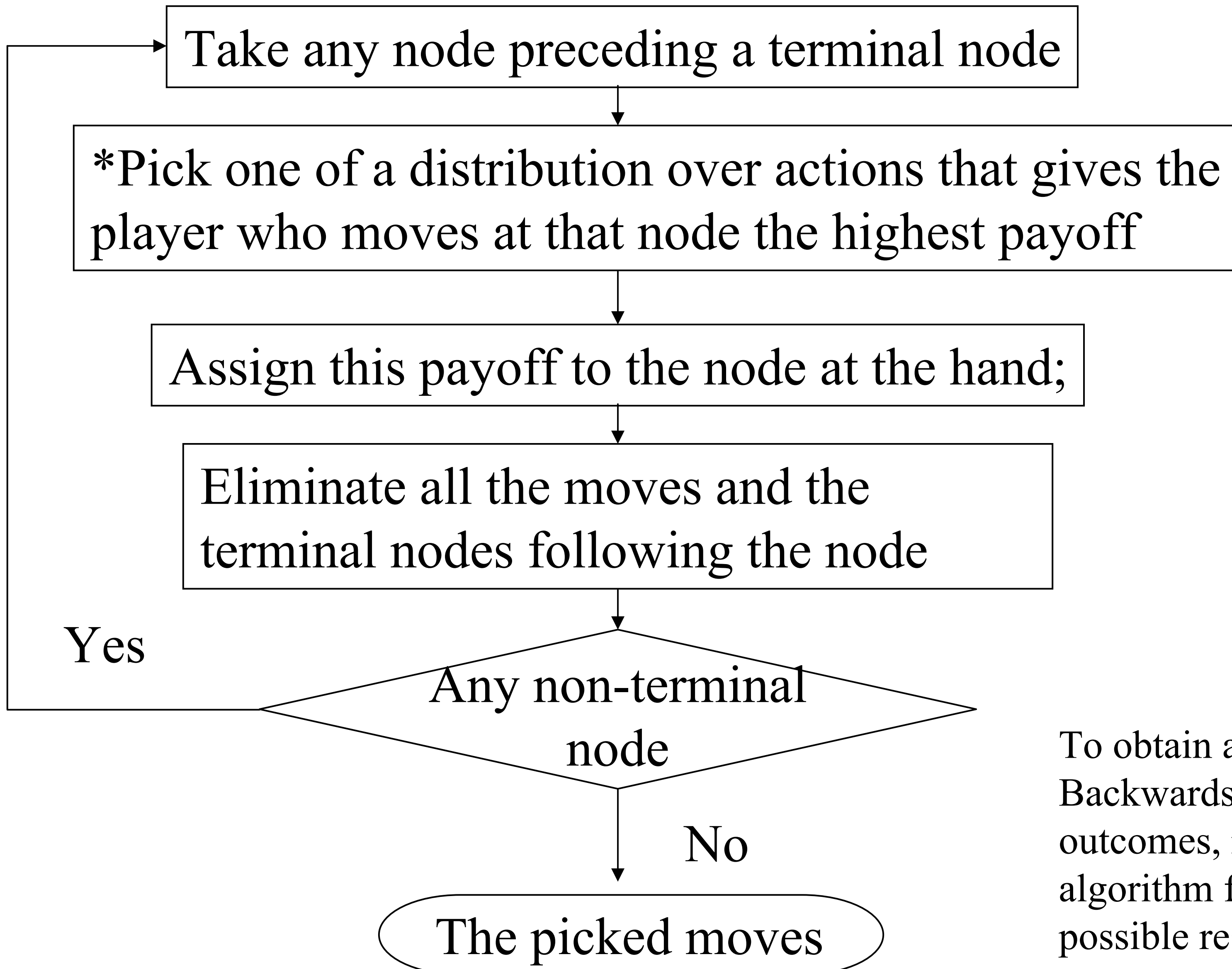
Note: Sequential rationality is required even at nodes that are ruled out by the players own strategy.

In Finite Ext. Form Games of Perfect Information:

CK of Seq. Rationality=>Backwards Induction



Backwards Induction (for finite & perfect info. games)



To obtain all Backwards Induction outcomes, repeat this algorithm for all possible resolutions of step *.

Subgame Perfection

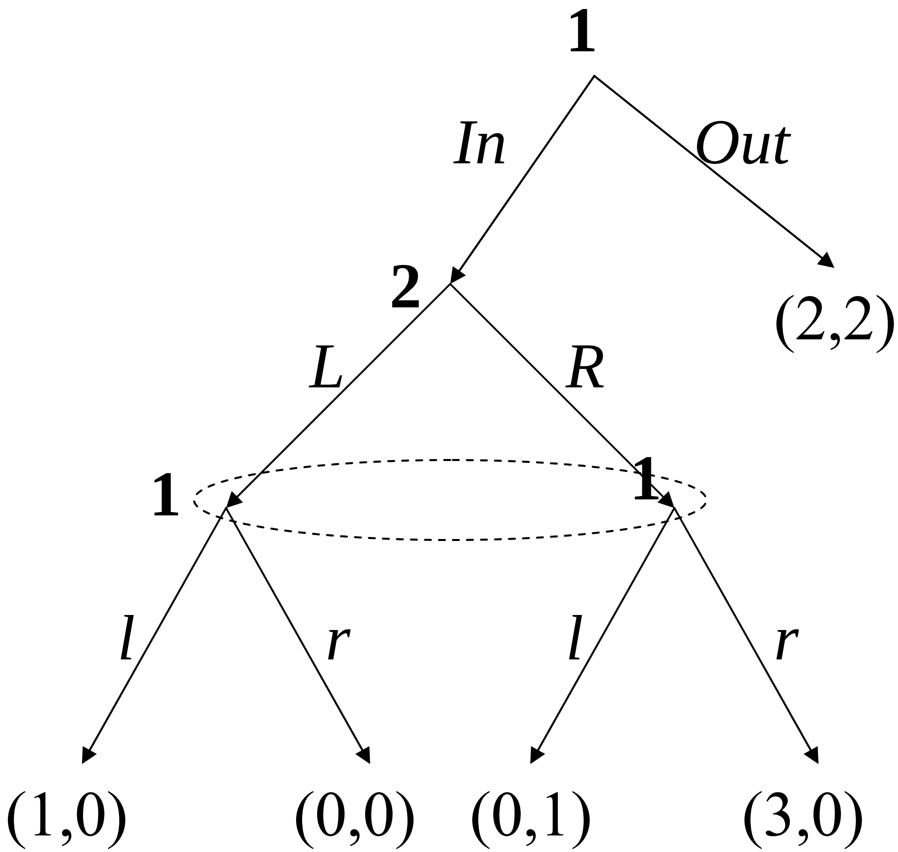
A **subgame** is part of a game that can be considered as a game itself: a history h **defines a subgame**, if $\forall I \in \mathcal{I}$ s.t. there is $(h, h') \in I$, we have that any $\tilde{h} \in I$ can be written as $\tilde{h} = (h, h'')$.

A profile of behavioral strategies β^* is a **Subgame Perfect (Nash) Equilibrium** if for any h that defines a subgame of Γ, any $i \in N$, and β_i:

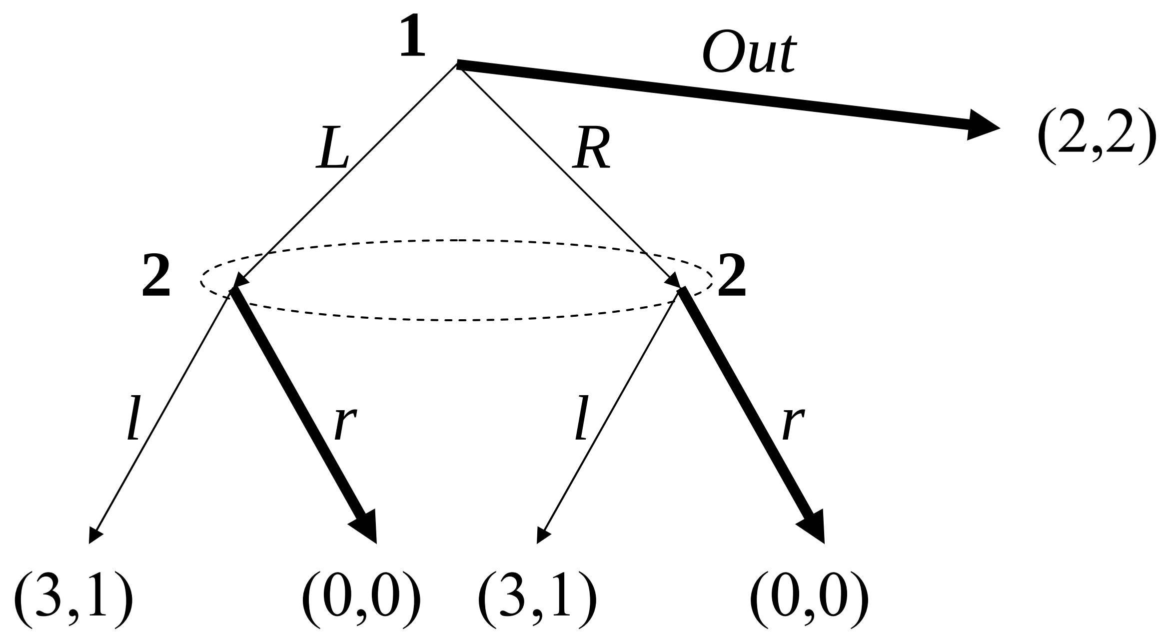
$$u_i(\mathcal{O}(\beta^*|h)) \geq u_i(\mathcal{O}(\beta_i, \beta^*{-}i|h)).$$

Theorem (Kuhn, 1953) In finite perfect information games backwards induction outcomes correspond exactly to subgame perfect equilibria.

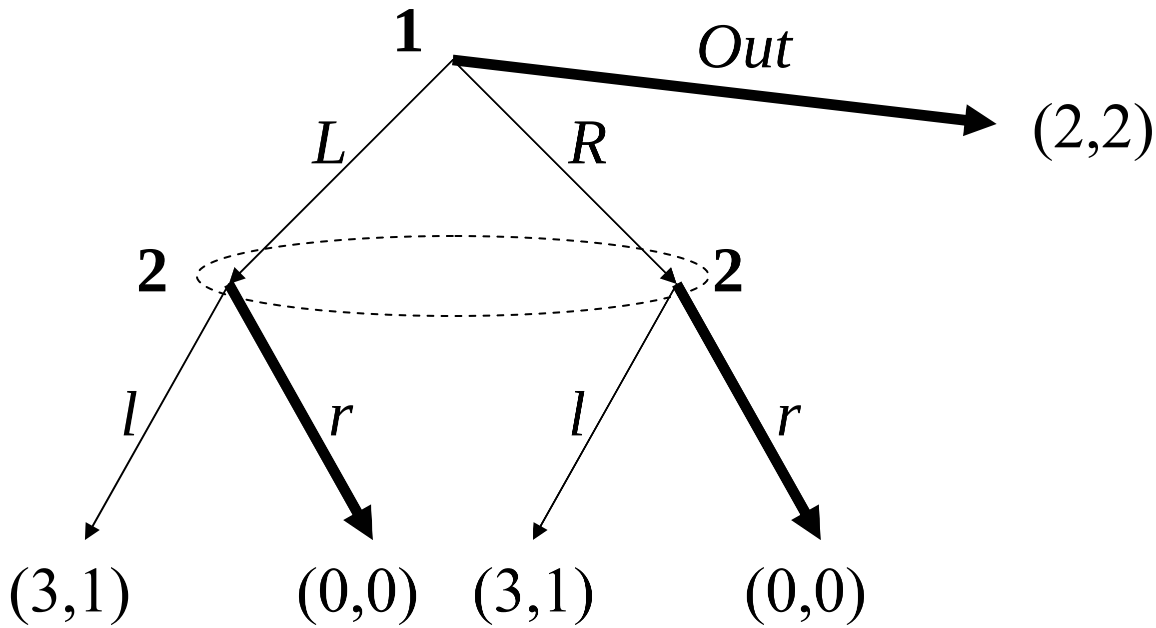
The subgame perfect equilibrium



What is wrong with this equilibrium?



What if players tremble?



Sequential Equilibrium

An **assessment** is a pair (β, μ) where β is a behavioral strategy profile and $\mu(\cdot|I)$ is a probability distribution over $A(I)$ for each $I \in \mathcal{I}$.

An assessment (β, μ) is **sequentially rational** if for each i, $I_i \in \mathcal{I}_i$, and β_i':

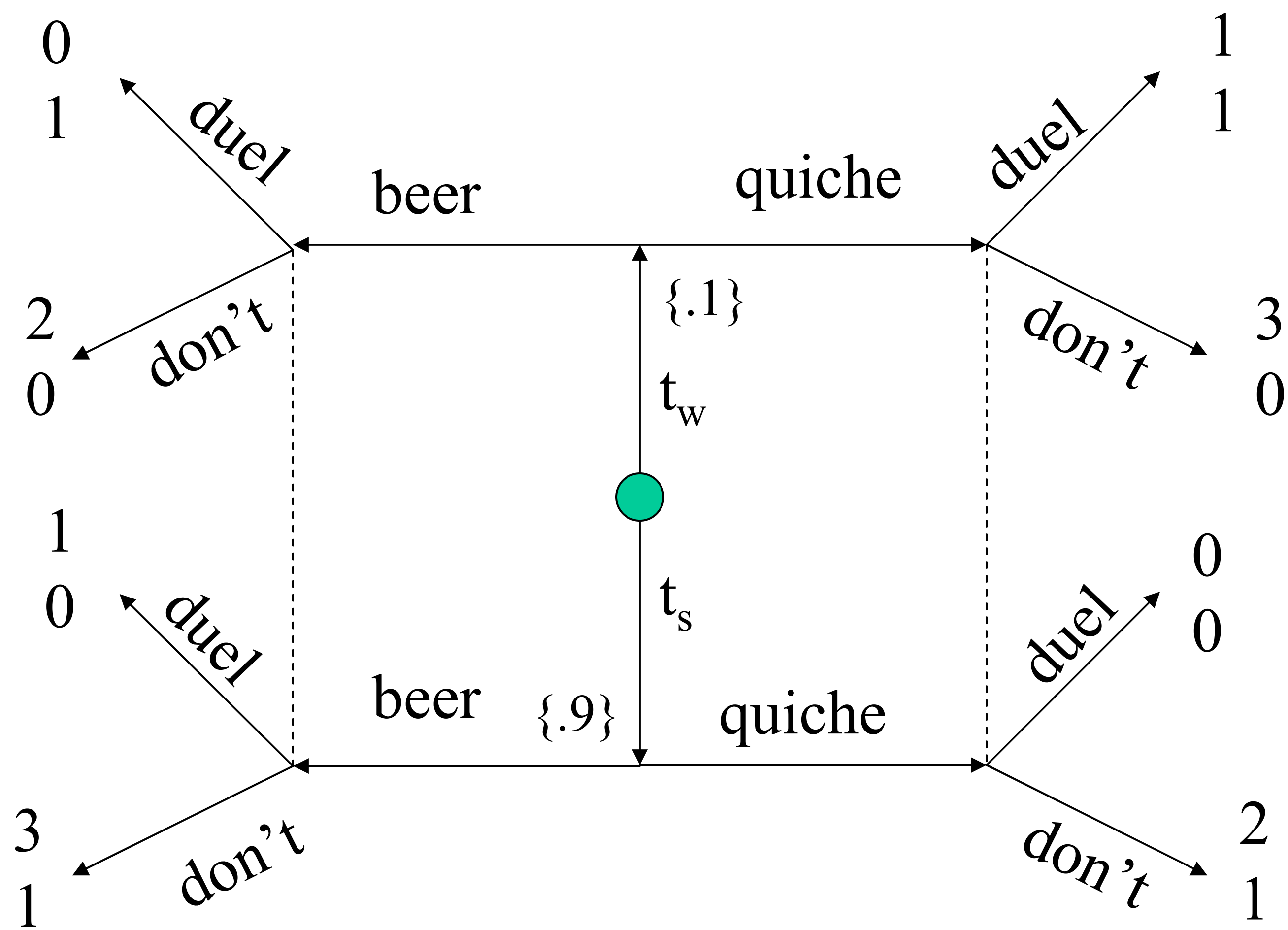
$$\mathbb{E}_{\mu(\cdot|I_i)} u_i(\mathcal{O}(\beta_i, \beta_{-i}|h)) \geq \mathbb{E}_{\mu(\cdot|I_i)} u_i(\mathcal{O}(\beta_i', \beta_{-i}|h)).$$

An assessment (β, μ) is **consistent** if there is a sequence (β^k, μ^k) of assessments s.t.

- $(\beta^k, \mu^k) \to (\beta, \mu)$,
- each β^k is completely mixed, and
- μ^k is derived from β^k using Bayes' rule.

A **sequential equilibrium** is a sequentially rational and consistent assessment.

Beer & Quiche



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