Learning 1
Replicator dynamics
&
Evolutionary stability

14.126 Game Theory

Haluk Ergin
Muhamet Yildiz

Road Map

1. Evolutionarily stable strategies
2. Replicator dynamics

Notation

- G = (S,A) a symmetric, 2-player game where
  - S is the strategy space;
  - $A_{i,j} = u_1(s_i,s_j) = u_2(s_j,s_i)$.
- x,y are mixed strategies; $u(x,y) = x^T Ay$. u(s,y).
- ax+(1-a)y.
- u(ax+(1-a)y,z) = au(x,z) +(1-a)u(y,z)
- u(x,ay+(1-a)z) = au(x,y) +(1-a)u(x,z)

ESS

Definition: A (mixed) strategy x is said to be *evolutionarily stable* iff, given any $y \neq x$, there exists $\varepsilon_y > 0$ s.t.

$$u(x,(1-\varepsilon)x+\varepsilon y) > u(y,(1-\varepsilon)x+\varepsilon y)$$

for each ε in $(0,\varepsilon_y]$.

- Each player is endowed with a (mixed) strategy.
- Assumes that population is a state
- Asks whether a strategy (state) is robust to evolutionary pressures.
- Disregards effects on future actions.

Alternative Definition

Fact: x is evolutionarily stable iff, $\forall y \neq x$,

1. $u(x,x) \geq u(y,x)$, and
2. $u(x,x) = u(y,x) \Rightarrow u(x,y) > u(y,y)$.

<u>Proof:</u> Define

$$F(\varepsilon,y) = u(x,(1-\varepsilon)x+\varepsilon y) - u(y,(1-\varepsilon)x+\varepsilon y) = u(x-y,x+\varepsilon(y-x))$$

$$= u(x-y,x) + \varepsilon\, u(x-y,y-x).$$

Hawk-Dove game




(1-c,1-c)	(*2*,0)
(0,*2*)	(*1*,*1*)

1. $c < 1$
2. $c > 1$

Figure by MIT OCW.

ESS-NE

- If x is an ESS, then (x,x) is a Nash equilibrium.
- In fact, (x,x) is a proper equilibrium.

Rock-Scissors-Paper

	R	S	P
R	0,0	1,-1	-1,1
S	-1,1	0,0	1,-1
P	1,-1	-1,1	0,0

- Unique Nash Equilibrium (s*,s*) where

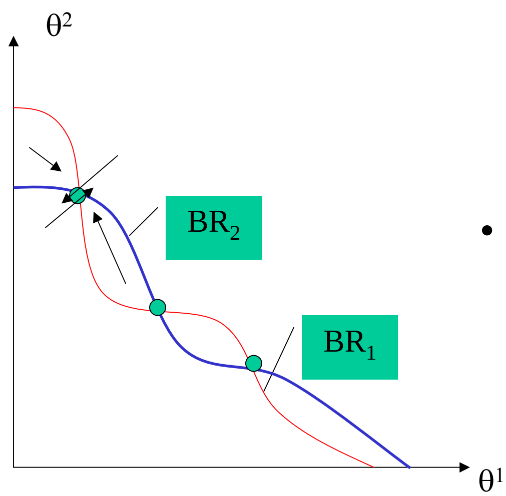
  s* = (1/3,1/3,1/3)
- s* is not ESS.

ESS in role-playing games

- Given (S^1,S^2,u_1,u_2), consider the symmetric game $(\underline{S},u)$ where
  - $\underline{S} = S^1 \times S^2$;
  - $u(\underline{x},\underline{y}) = [u_1(x_1,y_2)+ u_2(x_2,y_1)]/2\ \forall \underline{x}= (x_1,x_2),\ \underline{y}= (y_1,y_2) \in \underline{S}$.

Theorem: $\underline{x}$ is an ESS of $(\underline{S},u)$ iff $\underline{x}$ is a strict Nash equilibrium of (S^1,S^2,u_1,u_2).

Replicator Dynamics



- Steady state: $F(\theta) = \theta$
- A steady state θ of F is *stable* iff, $\forall$neighborhood U of θ, $\exists$nbrd U_1 s.t., if $\theta_0 \in U_1$, $F_t(\theta_t) \in U$ $\forall t>0$.
- A steady state θ of F is *asymptotically stable* iff, it is stable and, if $\theta_0 \in U$, $\lim_t F_t(\theta_t) = \theta$.

Replicator dynamics

- $p_i(t)$ = #people who plays s_i at t;
- $p(t)$ = total population at t.
- $x_i(t) = p_i(t)/p(t)$; $x(t) = (x_1(t),\ldots, x_k(t))$.
- $u(x,x) = \Sigma_i\, x_i u(s_i,x)$.
- Birthrate for s_i at t is $\beta + u(s_i,x(t))$.
- $\dot{p}_i = [\beta + u(s_i,x) - \delta]p_i$
- $\dot{p} = [\beta + u(x,x) - \delta]p$
- $\dot{x}_i = [u(s_i,x) - u(x,x)]x_i$
- $\dot{x}_i = u(s_i - x, x)x_i$

Example

- Consider (S,A) where $A = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}$
- $u(s_1,x) = a_1x_1$;
- $u(x,x) = (x_1,x_2)A(x_1,x_2)^T = a_1x_1^2 + a_2x_2^2$
- $u(s_1-x,x) = a_1x_1 - a_1x_1^2 - a_2x_2^2 = a_1x_1(1-x_1) - a_2x_2^2 = a_1x_1x_2 - a_2x_2^2 = (a_1x_1 - a_2x_2)x_2$
- $\dot{x}_1 = (a_1x_1 - a_2x_2)x_1x_2$

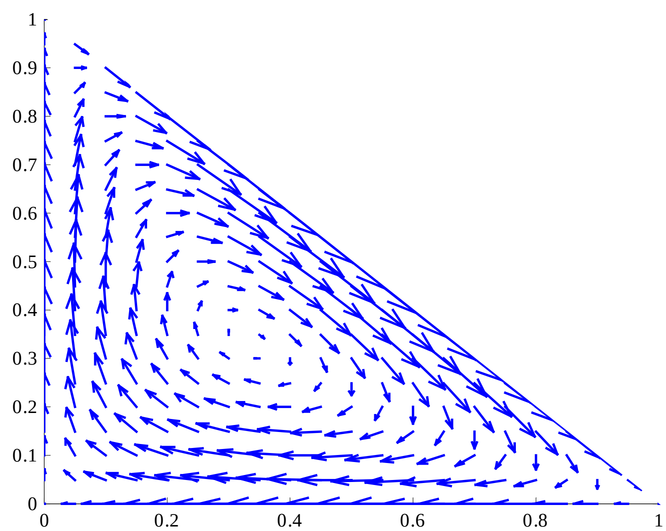
Observations

- $$\frac{d}{dt}\left[\frac{x_i}{x_j}\right] = \frac{\dot{x}_i}{x_j} - \frac{x_i}{x_j}\frac{\dot{x}_j}{x_j} = [u(s_i,x) - u(x,x)]\frac{x_i}{x_j} - \frac{x_i}{x_j}[u(s_j,x) - u(x,x)]\frac{x_j}{x_j}$$
  $$= [u(s_i,x) - u(s_j,x)]\frac{x_i}{x_j}$$
- If u becomes u = au+b, then Replicator dynamics becomes
  $$\dot{x}_i = \underline{u}(s_i - x, x)x_i = au(s_i - x, x)x_i$$

Examples

- Replicator dynamics in prisoners' dilemma
- Replicator dynamics in chicken
- Replicator dynamics in the battle of the sexes.

RD in Rock-Scissors-Paper



Rationalizability

- $\xi(.,x_0)$ is the solution to replicator dynamics starting at x_0.

Theorem: If a pure strategy i is strictly dominated (by y), then $\lim_t \xi_i(t,x_0) = 0$ for any interior x_0.

Proof: Define $v_i(x) = \log(x_i) - \Sigma_j y_j \log(x_j)$. Then,

$$\frac{dv_i(x(t))}{dt} = \frac{\dot{x}_i}{x_i} - \sum_j y_j \frac{\dot{x}_j}{x_j} = u(s_i - x, x) - \sum_j y_j u(s_j - x, x) = u(s_i - y, x).$$

Hence, $v_i(x(t)) \to -\infty$., i.e., $x_i(t) \to 0$.

Theorem: If i is not rationalizable, then $\lim_t \xi_i(t,x_0) = 0$ for any interior x_0.

Theorems

Theorem: Every ESS x is an asymptotically stable steady state of replicator dynamics.

(If the individuals can inherit the mixed strategies, the converse is also true.)

Theorem: If x is an asymptotically stable steady state of replicator dynamics, and can be reached from an interior x_0, then (x,x) is a perfect Nash equilibrium.