Interactive Epistemology –IV

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Road Map

1. Epistemic Foundations of Nash Equilibrium
2. Universal Type Space

Foundations of Nash Equilibrium

- Conjecture of a player i is

$$\phi^i = \operatorname*{marg}_{A_{-i}} p(.\,|\,I_i(\omega))$$

- Conjecture of a player i about j is marginal of ϕ^i on A_j.

Theorem

Consider a two person game (A,u). Let $(\Omega,\mathfrak{I}_i,p_i,\mathbf{u},s_i)_{i=1,2}$ be a model, and $\psi = (\psi^1,\psi^2)$ be a pair of conjectures. Assume that at some state ω it is mutually known that

1. $\mathbf{u}(\omega) = u$,
2. The players are rational,
3. $\phi = \psi$.

Then, (ψ^2, ψ^1) is a Nash equilibrium.

Proof (n=2)

- Take any a_i with $\psi^j(a_i) > 0$.
- $a_i = s_i(\omega')$ for some $\omega' \in I_j(\omega)$.
- $\phi^i(w') = \psi^i$.
- $a_i = s_i(\omega') \in BR_i(u; \phi^i(\omega')) = BR_i(u; \psi^i)$.
- QED

Theorem

Consider a game (A,u). Let $(\Omega,\mathfrak{I}_i,p_i,\mathbf{u},s_i)_{i=1,\ldots n}$ be a model, and ψ be an n-tuple of conjectures. Assume CPA and that with positive probability it is mutually known that

1. $\mathbf{u}(\omega) = u$,
2. The players are rational,

and it is common knowledge that $\phi = \psi$.

Then, given any i marginal of each ψ^j on A_i is same σ_i, and σ is a Nash equilibrium.

Universal Type space

- Θ = the basic space of uncertainty.
- N = {1,2,…,n} agents
- $\Delta(X)$ is the set of probability distributions on X, endowed with weak* topology.
- Define
  - $X_0 = \Theta$;
  - $X_1 = [\Delta(X_0)]^n \times X_0 = [\Delta(\Theta)]^n \times S$;
  - …
  - $X_k = [\Delta(X_{k-1})]^n \times X_{k-1}$
- Beliefs of i: $T_i^0 = \Pi_{k=1}^{\infty} \Delta(X_{k-1}) = \Delta(X_0) \times \Delta(X_1) \times \ldots$
- A type of i: $(t_i^1, t_i^2, \ldots, t_i^m, \ldots)$
- $\Omega' = \Theta \times T_1^0 \times T_2^0 \times \ldots \times T_n^0$

Coherence

Definition: T_i^1 = coherent beliefs of i, i.e.,

1. $\text{marg}_{X_{k-1}} t_i^{k+1`} = t_i^k$
2. marg of t_i^{k+1} on the ith copy of $\Delta(X_{k-1})$ is point mass at t_i^k.

Theorem: For each $t_i \in T_i^1$, there exists a belief $b_i[t_i] \in \Delta(\Theta \times T^0)$ such that

$$t_i^k = \text{marg}_{X_{k-1}} b_i^{`}[t_i] \qquad (\forall k).$$

Definition: $\beta_i[t_i] = \text{marg}_{\Theta \times T_{-i}^0} b_i^{`}[t_i] \in \Delta(\Theta \times T_{-i}^0)$

More Definitions

- A subspace $Y = Y_1 \times Y_2 \times \ldots \times Y_n \subset T^1$ is *belief-closed* iff

$$\mathrm{supp}(b_i[t_i]) \subset \Theta \times Y.$$

- **Universal Type Space:**

$$T = T_1 \times T_2 \times \ldots \times T_n$$

 is the largest belief-closed subspace of T^1.
- $\Omega = \Theta \times T$

A traditional Type Space

- $N = \{1,2\}$; $\Theta = R$;
- Each i gets a signal $x_i = \theta + \varepsilon_i$, where
- $(\theta,\varepsilon_1,\varepsilon_2)$ iid with $N(0,1)$.
- $t_1^1[x_1] =$
- $t_1^2[x_1] =$

Technical Theorems

- Topology on T:

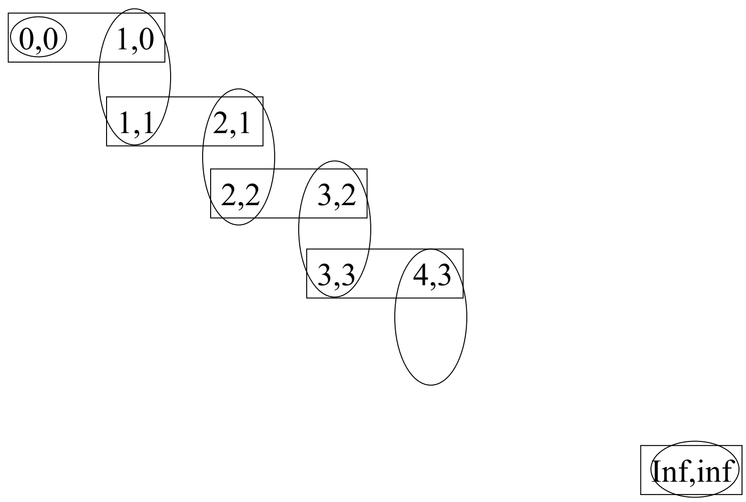
  $[t(m) \to t] \Leftrightarrow [t_i^k(m) \to t_i^k \ \forall i,k]$.
- $t_i(m) \to t_i \Leftrightarrow b_i[t_i(m)] \to b_i[t_i]$.
- If Θ is a complete, separable metric space, so is T.
- If Θ is compact, separable metric space, so is T.
- If Θ is compact, separable metric space, then $\forall t_i \in T_i$, $\exists t_i(m) \to t_i$ s.t. each $t_i(m) \in T_i[m]$ for some finite, belief-closed subspace T[m].

Electronic Mail Game

	A	B
A	3,3	0,2
B	2,0	2,2

	A	B
A	0,0	0,2
B	2,0	2,2

E-mail



Almost common knowledge

- $B_i^q(A) = \{ \omega \mid p_i(A|I_i(\omega)) \geq q \}$
- E is common q-belief at ω ⇔

$$\omega \in \bigcap_{n=1}^{\infty} \left(B_N^q\right)^n (E)$$

- E is q-evident ⇔ E is common q-belief at each ω in E.
- Robustness against incomplete information.