14.126 Game Theory

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Lecture 2:

Choice under Uncertainty
(continued)

Qualitative Probability

Extracting perceived relative likelihoods of events from choices:

$$\begin{pmatrix} \$100 & A \\ \$0 & A^c \end{pmatrix} \text{ or } \begin{pmatrix} \$100 & B \\ \$0 & B^c \end{pmatrix} \quad ?$$

$\mu : \mathcal{A} \to [0,1]$ is a **probability measure** if $\mu(S) = 1$ and $\mu(A \cup B) = \mu(A) + \mu(B)$ when $A \cap B = \emptyset$.

A binary relation $\succeq^*$ on $\mathcal{A}$ is a **qualitative probability** if:

(a) $\succeq^*$ is complete and transitive,

(b) $A \succeq^* \emptyset$ for any A,

(c) $S \succ^* \emptyset$, and

(d) If $A \cap C = B \cap C = \emptyset$, then: $A \succeq^* B \Leftrightarrow A \cup C \succeq^* B \cup C$.

Properties of Qualitative Probability

Proposition 1 *Let $\succeq^*$ be a qualitative probability, then the following are satisfied:*

(i) *If $A \supset B$ then $S \succeq^* A \succeq^* B \succeq^* \emptyset$.*

(ii) *If $A \sim^* B$ and $A \cap C = \emptyset$ then $A \cup C \succeq^* B \cup C$.*

(ii') *If $A \succ^* B$ and $A \cap C = \emptyset$ then $A \cup C \succ^* B \cup C$.*

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(iii) *If $A \sim^* B$, $C \sim^* D$, and $A \cap C = \emptyset$ then $A \cup C \succeq^* B \cup D$.*

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(iv) *If $A \sim^* B$, $C \sim^* D$, and $A \cap C = B \cap D = \emptyset$ then $A \cup C \sim^* B \cup D$.*

Can any qualitative probability be represented by a probability measure on S?

Savage's construction requires the additional condition:

(e) $A \succ^* B$ implies that there is a partition $C_1, \ldots, C_n$ of S such that $A \succ^* B \cup C_i$ for any i.

Proposition 2 *Let $\succeq^*$ be a qualitative probability satisfying (e). Then, if $A \succ^* \emptyset$, there is $B \subset A$ s.t. $B \sim^* A \setminus B$.*

μ is **nonatomic** if for any A with $\mu(A) > 0$ and $\lambda \in [0, 1]$, there is $B \subset A$ such that $\mu(B) = \lambda\mu(A)$.

Theorem 1 *$\succeq^*$ satisfies (a)–(e) iff there is a nonatomic probability measure $\mu \colon \mathcal{A} \to [0, 1]$ s.t.:*

$$A \succeq^* B \;\Leftrightarrow\; \mu(A) \geq \mu(B).$$

Moreover μ is unique.

Proof of Sufficiency

Suppose that $\succeq^*$ satisfies (a)–(e).

Step 1: Construct Nested Equipartitions:

Set $A_1^0 = S \succ^* \emptyset$. Inductively for $n \geq 1$:

A_{2k-1}^n, A_{2k}^n is a partition of A_k^{n-1} s.t. $A_{2k-1}^n \sim^* A_{2k}^n \succ^* \emptyset$.

$A_1^n, \ldots, A_{2^n}^n$ is a partition of S s.t. $A_i^n \sim^* A_j^n$ for any i, j.

Note that for distinct $i_1, \ldots, i_k$, and distinct $j_1, \ldots, j_m$:

$$\bigcup_{l=1}^{k} A_{i_l}^n \succeq^* \bigcup_{l=1}^{m} A_{j_l}^n \quad \Leftrightarrow \quad k \geq m.$$

Step 2: Approximate Probabilities:

$k(A, n) := \max\{k = 0, 1, \ldots, 2^n \mid A \succeq^* \bigcup_{i=1}^{k} A_i^n\}$

$r(A, n) := \min\{r = 0, 1, \ldots, 2^n \mid \bigcup_{i=1}^{r} A_i^n \succeq^* A\}$

Set $\mu(A) = \lim_{n\to\infty} k(A, n)/2^n = \lim_{n\to\infty} r(A, n)/2^n$.

Step 3: Verify that μ is additive.

Step 4: Verify that μ is non-atomic.

Step 5: Verify that μ represents $\succeq^*$.

Representation of Savage Acts

If $A_1, \ldots, A_n$ is a partition of S and $f_1, \ldots, f_n \in F$, the act:

$$\begin{pmatrix} f_1 & A_1 \\ \vdots & \vdots \\ f_n & A_n \end{pmatrix} \text{ is defined by } \begin{pmatrix} f_1 & A_1 \\ \vdots & \vdots \\ f_n & A_n \end{pmatrix}(s) = \begin{cases} f_1(s) & \text{if } s \in A_1 \\ \vdots & \vdots \\ f_n(s) & \text{if } s \in A_n \end{cases}.$$

For any $f \in F$, there is a partition $A_1, \ldots, A_n$ of S and $x_1, \ldots, x_n \in X$ such that:

$$f = \begin{pmatrix} x_1 & A_1 \\ \vdots & \vdots \\ x_n & A_n \end{pmatrix}.$$

$A^c = S \setminus A$.

$f =_A g$ means $f(s) = g(s)$ for any $s \in A$.

Savage's Six Axioms

Axiom 4.2.1. (Preference) $\succeq$ is a preference on F.

Axiom 4.2.2. (Non-degeneracy) There are $x^*, x_* \in X$ such that $x^* \succ x_*$.

Axiom 4.2.3. (Monotonicity) For any non-null event A, if f and g are two acts such that $f =_{A^c} g$ and $f =_A x$ and $g =_A y$, then $f \succeq g$ if and only if $x \succeq y$.

If A_1 is non-null then: $x \succeq y \;\Leftrightarrow\; \begin{pmatrix} x & A_1 \\ z_2 & A_2 \\ \vdots & \vdots \\ z_n & A_n \end{pmatrix} \succeq \begin{pmatrix} y & A_1 \\ z_2 & A_2 \\ \vdots & \vdots \\ z_n & A_n \end{pmatrix}.$

Axiom 4.2.4. (Continuity) For any $f, g \in F$ with $f \succ g$ and $x \in X$, there is a partition $A_1, \ldots, A_n$ of S such that:

(i) For any i, if $f' =_{A_i^c} f$ and $f' =_{A_i} x$ then $f' \succ g$,

(ii) For any i, if $g' =_{A_i^c} g$ and $g' =_{A_i} x$ then $f \succ g'$.

Axiom 4.2.5. (Comparative Probability) For any two events A and B and $x, y, x', y' \in X$ such that $x \succ y$ and $x' \succ y'$:

$$\begin{pmatrix} x & A \\ y & A^c \end{pmatrix} \succeq \begin{pmatrix} x & B \\ y & B^c \end{pmatrix} \Leftrightarrow \begin{pmatrix} x' & A \\ y' & A^c \end{pmatrix} \succeq \begin{pmatrix} x' & B \\ y' & B^c \end{pmatrix}.$$

Axiom 4.2.6. (Sure-Thing Principle) For any acts $f, g, f', g' \in F$ and event A such that $f =_A f'$ and $g =_A g'$, $f =_{A^c} g$, and $f' =_{A^c} g'$, we have $f \succeq g$ if and only if $f' \succeq g'$.

Axiom 4.2.6: Sure-Thing Principle

Visual Restatement 1:

$$\begin{pmatrix} f & A \\ h & A^c \end{pmatrix} \succeq \begin{pmatrix} g & A \\ h & A^c \end{pmatrix} \Leftrightarrow \begin{pmatrix} f & A \\ \widehat{h} & A^c \end{pmatrix} \succeq \begin{pmatrix} g & A \\ \widehat{h} & A^c \end{pmatrix}.$$

Visual Restatement 2:

$$\begin{pmatrix} x_1 & A_1 \\ \vdots & \vdots \\ x_n & A_n \\ z_1 & C_1 \\ \vdots & \vdots \\ z_k & C_k \end{pmatrix} \succeq \begin{pmatrix} y_1 & B_1 \\ \vdots & \vdots \\ y_m & B_m \\ z_1 & C_1 \\ \vdots & \vdots \\ z_k & C_k \end{pmatrix} \Leftrightarrow \begin{pmatrix} x_1 & A_1 \\ \vdots & \vdots \\ x_n & A_n \\ \widehat{z}_1 & C_1 \\ \vdots & \vdots \\ \widehat{z}_k & C_k \end{pmatrix} \succeq \begin{pmatrix} y_1 & B_1 \\ \vdots & \vdots \\ y_m & B_m \\ \widehat{z}_1 & C_1 \\ \vdots & \vdots \\ \widehat{z}_k & C_k \end{pmatrix}.$$

Subjective EU

Given μ and $f \in F$, define the lottery $p_f^\mu \in P$ by:

$$p_f^\mu(x) = \mu\left(f^{-1}(x)\right), \qquad x \in X.$$

(p_f^μ: the distribution over prizes induced by f and μ.)

If $u\colon X \to \mathbb{R}$, then:

$$\mathbb{E}_\mu[u \circ f] = \sum_{x \in X} u(x) p_f^\mu(x) \qquad f \in F.$$

Theorem 3: *(Savage, 1954) $\succeq$ satisfies 4.2.1–4.2.6 iff there exist a non-atomic probability measure μ on S and a non-constant utility function $u\colon X \to \mathbb{R}$ s.t.:*

$$f \succeq g \;\Leftrightarrow\; \mathbb{E}_\mu[u \circ f] \geq \mathbb{E}_\mu[u \circ g] \qquad f, g \in F.$$

Moreover μ is unique and u is unique up to a positive affine transformation.

Sketch of the Sufficiency Proof

Suppose that $\succeq$ satisfies 4.2.1–4.2.6.

Define $\succeq^*$ by:

$A \succeq^* B \;\Leftrightarrow\; \begin{pmatrix} x & A \\ y & A^c \end{pmatrix} \succeq \begin{pmatrix} x & B \\ y & B^c \end{pmatrix}$ for any x, y such that $x \succ y$.

Lemma 1 $\succeq^*$ *satisfies (a)–(e). Therefore, there exists a unique non-atomic probability measure* μ *s.t.:*

$$A \succeq^* B \;\Leftrightarrow\; \mu(A) \geq \mu(B).$$

Lemma 2 $p_f^\mu = p_g^\mu \Rightarrow f \sim g$.

Lemma 3 *For any* $p \in P$, *there is* $f \in F$ *such that* $p = p_f^\mu$.

Define a binary relation $\succeq^S$ over P by:

$$p_f^\mu \succeq^S p_g^\mu \Leftrightarrow f \succeq g.$$

Lemma 4 *The $\succeq^S$ is a well defined preference relation on P that satisfies Independence and vNM-Continuity.*

By vNM Theorem, there is a $u\colon X \to \mathbb{R}$ s.t.:

$$p \succeq^S q \Leftrightarrow \sum_{x \in X} p(x)u(x) \geq \sum_{x \in X} q(x)u(x) \qquad p, q \in P.$$

Hence:

$$f \succeq g \;\Leftrightarrow\; p_f^\mu \succeq^S p_g^\mu \Leftrightarrow \; \mathbb{E}_\mu[u \circ f] \geq \mathbb{E}_\mu[u \circ g].$$

Limitations of Savage (State space must be objective, state-dependent utility excluded, actions can not affect probabilities)