Interactive Epistemology

14.126 Game Theory
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Road Map

1. Formalizing knowledge for a single person
2. Common Knowledge
   1. Among two individuals
   2. Among many individuals
   3. Agreement Theorem
3. Constructing a universal state space
4. Syntactic and semantic approach
5. No-trade theorem

Basics

- ω = a state, a complete description of the world;
- Ω = the set of all states;
- E = an event, a subset of Ω;
  - $E \subseteq F$ = E implies F
  - $E \cap F$ = E and F
  - $E \cup F$ = E or F
- $\mathfrak{E}$ = the set of all events

Equivalent Formalizations of knowledge

- **Knowledge function:** κ from Ω to some space; the agent knows the value of κ, i.e., he knows $\kappa(\omega)$, but not ω.
- **Information Function:** $I(\omega) = \{\omega' \in \Omega | \kappa(\omega) = \kappa(\omega')\}$
  - $\omega \in I(\omega)$;
  - If $I(\omega) \cap I(\omega') \neq \emptyset$, then $I(\omega) = I(\omega')$

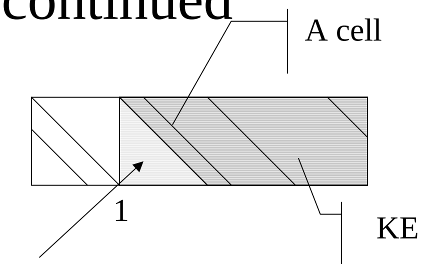
- p = price of bread ($p \geq 0$);
- e = noise ($e \in [0,1]$)
- $\omega = (p,e)$
- He gets a noisy signal $\kappa = p+e$.
- $I(p,e) = \{(p',e') | p'+e' = p+e\}$

Knowledge, continued

- **Information partition:**
- $\mathfrak{I} = \{I(\omega) | \omega \in \Omega\}$
- **Knowledge u(niversal)field:**

  $\mathfrak{K}$ = all possible unions of cell in $\mathfrak{I}$.
- **Knowledge operator**:

  $K : \mathfrak{E} \to \mathfrak{E}$
- $KE = \{\omega \in \Omega | I(\omega) \subseteq E\}$
  - $KE = \cup_{F \in \mathfrak{I}, F \subseteq E} F$
  - KE is the largest member of $\mathfrak{K}$ included in E.



- $E = \{(p,e) | \ p > 1\}$
- $I(p,e) \subseteq E$ iff $p + e > 2$.
- $KE = \{(p,e) | p + e > 2\}$

Knowledge operator

1. $KE \subseteq E$
2. $E \subseteq F$ implies $KE \subseteq KF$
3. $KE \subseteq KKE$
4. $\sim KE \subseteq K \sim KE$

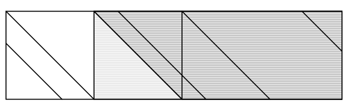
- $K(E \cap F) = K(E) \cap K(F)$.
- $KE = KKE$;
- $\sim KE = K \sim KE$.
- $K\Omega = \Omega$.

1. If I know something it must be true.
2. If E logically implies F, and if I know that E is true, then I know that F is true.
3. If I know something, I know that I know that.
4. If I don’t know something, I know that I don’t know that.

Associated!

Common knowledge (2 person)

- $i = 1,2$; K_i, I_i,etc.
- CKE := $K_1E \cap K_2E \cap K_1K_2E \cap K_2K_1E \cap K_1K_2K_1E \cap K_2K_1K_2E \cap K_1K_2K_1K_2E \cap K_2K_1K_2K_1E \ldots$



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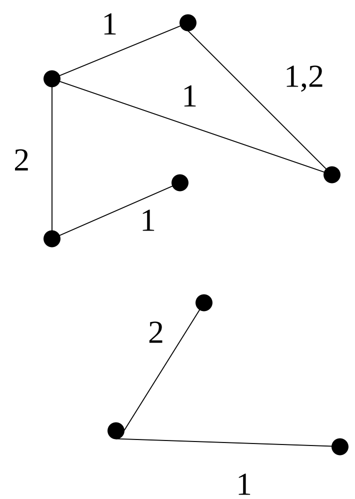
- $\kappa_1 = p+e$; $\kappa_2 = p$.
- $I_1(p,e) = \{(p',e')|p'+e'=p+e\}$
- $I_2(p,e) = \{(p',e')|p'=p\}$
- $E = \{(p,e)| \ p > 1\}$
- $K_1E =\{(p,e)|p + e > 2\}$
- $K_2E =\{(p,e)|p > 1\} = E$;
- $K_1K_2E = K_1E$
- $K_2K_1E =\{(p,e)|p > 2\}$ (why?)
- $K_1K_2K_1E = \{(p,e)|p + e > 3\}$
- CKE =

Theorems about CK (2-person)

1. $CKE \subseteq E$
2. $K_iCKE = CKE$
3. $E \subseteq F \Rightarrow CKE \subseteq CKF$
4. CKE is the largest event F with $F \subseteq E$ and $K_1F = K_2F = F$.
5. CK is a knowledge operator associated with $\mathfrak{K}_1 \cap \mathfrak{K}_2$.
6. CKE = CKCKE ; ~CKE = CK~CKE
7. $CKE\Omega = \Omega$.

A Graphical approach



- Ω = all the nodes;
- Two nodes are in the same sell of i iff they are connected by an edge indicated by i;
- CK partition = largest connected subgraphs.

Common Coarsening $\mathfrak{I}_1 \wedge \mathfrak{I}_2$

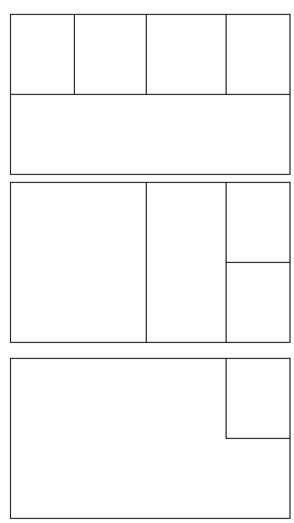
- $\mathfrak{I}_1 \wedge \mathfrak{I}_2$ is the finest partition that is coarser than both I_1 and I_2, associated with $\mathfrak{K}_1 \cap \mathfrak{K}_2$.

$$I_{1,2}(\omega) = \bigcap_{\omega \in F \in \mathfrak{K}_1 \cap \mathfrak{K}_2} F$$

- $I_1(\omega) \cup I_2(\omega) \subseteq I_{1,2}(\omega)$.
- $I_{1,2}(\omega)$ can be written as the union of some cells in $\mathfrak{I}_1$.
- $\omega' \in I_{1,2}(\omega)$ iff $\exists \omega_0, \omega_1, \ldots, \omega_n$, $\exists i(0), i(1), \ldots, i(n)$ s.t. $\omega_0 = \omega$, $\omega_n = \omega'$,

  $\omega_k \in I_{i(k-1)}(\omega_{k-1})$

  for all $k > 0$.



Common knowledge (n person)

- $i = 1,2,\ldots,n$; K_i, I_i, etc.
- $K^1 := \cap_i K_i$;
- $K^m := (K^1)^m$;
- $CKE := \cap_m K^m$

1. $CKE \subseteq E$
2. $K_i CKE = CKE$
3. $E \subseteq F \Rightarrow CKE \subseteq CKF$
4. CK is a knowledge operator associated with $\mathfrak{K}_1 \cap \mathfrak{K}_2 \cap \ldots \cap \mathfrak{K}_n$.
5. CKE is the largest event F with $F \subseteq E$ and $K_1 F = K_2 F = \ldots = K_n F = F$.
6. $CKE = CKCKE$; $\sim CKE = CK\sim CKE$
7. $CK E\Omega = \Omega$.

Common Coarsening – same

- $\mathfrak{I}_1 \wedge \mathfrak{I}_2 \wedge \ldots \wedge \mathfrak{I}_n$ is the finest partition that is coarser than all $\mathfrak{I}_1 \ldots \mathfrak{I}_n$, associated with $\mathfrak{K}_1 \cap \mathfrak{K}_2 \cap \ldots \cap \mathfrak{K}_n$.

$$I_{1,\ldots,n}(\omega) = \bigcap_{\omega \in F \in \mathfrak{K}_1 \cap \ldots \cap \mathfrak{K}_n} F$$

- $I_{1,\ldots,n}(\omega)$ can be written as the union of some cells in $\mathfrak{I}_1$.
- $\omega' \in I_{1,\ldots,n}(\omega)$ iff $\exists \omega_0, \omega_1, \ldots, \omega_n$, $\exists i(0), i(1), \ldots, i(n)$ s.t. $\omega_0 = \omega$, $\omega_n = \omega'$,

$$\omega_k \in I_{i(k-1)}(\omega_{k-1})$$

for all $k > 0$.

Agreement Theorem

- B = a finite set of decisions b;
- d : E \ {∅} → B, a decision rule;
- d satisfies the sure-thing principle: if $E = \cup_a J_a$ where $\{J_a\}$ is a family of disjoint sets, and if $d(J_a) = b$ at each a, then d(E) = b.

Theorem: For each i, define $d_i \Omega \rightarrow B$ by $d_i(\omega)=d(I_i(\omega))$. Then,

$$CK(d_1=b) \cap CK(d_2=c) \neq \emptyset \Rightarrow b = c.$$

Proof: Assume: $\omega \in E := CK(d_1=b) \cap CK(d_2=c)$.

1. $\exists \{J_a\} \subseteq \mathfrak{I}_1$ s.t. $E = \cup_a J_a$.
2. $d(J_a) = b$ for each a.
3. d(E) = b.

Application – agreeing to disagree

- A = a fixed event
- B = [0,1]
- d(E) = P(A|E), conditional probability w.r.t. a common prior P given E.
- Bayes' rule => d satisfies the sure-thing principle.
- Agreement Theorem => if it is common knowledge at w that $P(A|I_1(\omega)) = b$ and $P(A|I_2(\omega)) = b$, then b = c.

No-trade Theorem

- $\omega = (x,z)$; $z = (z_1,\ldots,z_n)$, i observes z_i, owns $e_i(x)$.
- $y : \Omega \to B$, $y \in Y$.
- $u_i(y_i(\omega);x) := \underline{u}_i(y_i(\omega) + e_i(x);x)$.

Lemma: Assume that y is Pareto-optimal, and y' and $A \subseteq \Omega$ are s.t. $E[u_i(y_i'(\omega);x)|A] \geq E[u_i(y_i(\omega);x)|A]$ and $Prob(A) > 0$. Then, $E[u_i(y_i'(\omega);x)|A] = E[u_i(y_i(\omega);x)|A]$. If each u_i is strictly concave, then y' = y on A.

Proof: Define y* = [y' on A; y on ~A]. Apply the sure-thing principle. Then, $E[u_i(y_i^*(\omega);x)] \geq E[u_i(y_i(\omega);x)]$. If $E[u_i(y_i'(\omega);x)|A] > E[u_i(y_i(\omega);x)|A]$, then $E[u_i(y_i^*(\omega);x)] > E[u_i(y_i(\omega);x)]$.

No-trade Theorem

Theorem: Assume that $\underline{y} = 0$ is Pareto-optimal, and $Prob(I_{1,\ldots,n}(\omega)) > 0$. If it is common knowledge at ω that y is feasible and each i weakly prefers y to $\underline{y}$, then each is indifferent between y and $\underline{y}$. If each agent is strictly risk averse, then $y = \underline{y}$.

Proof: Take $A = I_{1,\ldots,n}(\omega)$ in the lemma.

An equilibrium example

- $\Omega = \{\omega_1,\omega_2\}$.
- 2 players, i = 1,2
- $I_1 = \{\{\omega_1\},\{\omega_2\}\}$;
- $I_2 = \{\omega_1,\omega_2\}$;

- Asset 1 gives
  - \$–1 at ω_2.
  - \$10 at ω_1 and
- Asset 2 gives
  - \$–1 at ω_1, and
  - \$1 at ω_2.

Universal state-space

- X = an alphabet of letters x, y, z, …
- Formulas: finite strings of symbols s.t.
  - Every letter is a formula;
  - If f and g are formulas, so is (f)OR(g);
  - If f is a formula, so are NOT(f) and k_i.
- A *list* L (of formulas) is *logically closed* iff

  $[f \in L \;\&\; (f \Rightarrow g) \in L] \Rightarrow g \in L$;
- *Epistemically closed* iff

  $f \in L \Rightarrow k_i f \in L$.

- A state ω is any logically closed list s.t.
  - $f \in \omega \Leftrightarrow NOT(f) \notin \omega$;
  - ω includes all the "tautologies."
- Ω = the set of all states.
- $\kappa_i(w) = \{k_i(f) | k_i(f) \in \omega\}$

Tautologies

The set of tautologies is the smallest logically and epistemically closed list that contains all:

- (f OR f) => f
- f => (f OR g)
- (f OR g) => (g OR f)
- (f => g) => ((h OR f) => (h OR g))
- k_i(f) => f
- (k_i(f=>g)) => (k_i(f)=>k_i(g))
- k_i(f) => k_i(k_i(f))
- NOT(k_i(f)) => k_i(NOT(k_i(f)))

Some theorems

- $E_f := \{\omega \in \Omega | f \in \omega\}$
- $\sim E_f = E_{NOT(f)}$
- $E_f \cup E_g = E_{(f\ OR\ g)}$
- $E_f \cap E_g = E_{(f\ \&\ g)}$
- $K_i E_f = E_{ki(f)}$
- $E_f \subseteq E_g$ $\Leftrightarrow$ [f => g is a tautology]