Lecture 3
Representation of Games

14.12 Game Theory

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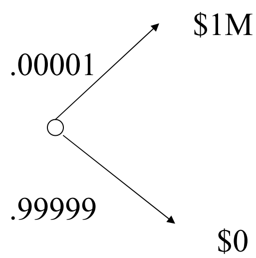
Road Map

1. Cardinal representation – Expected utility theory
2. **Quiz**
3. Representation of games in strategic and extensive forms
4. Dominance; dominant-strategy equilibrium

Cardinal representation – definitions

- Z = a finite set of consequences or prizes.
- A lottery is a probability distribution on Z.
- P = the set of all lotteries.
- A lottery:



Cardinal representation

- Von Neumann-Morgenstern representation:

A lottery (in P)

Expected value of u under p

$$p \succeq q \Leftrightarrow \underbrace{\sum_{z \in Z} u(z)\, p(z)}_{U(p)} \geq \underbrace{\sum_{z \in Z} u(z) q(z)}_{U(q)}$$

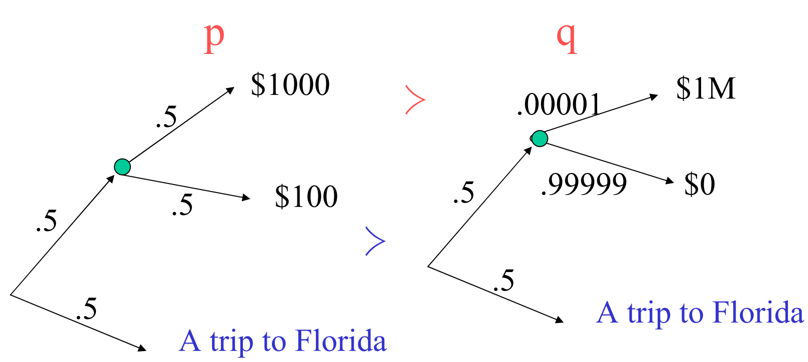
VNM Axioms

Axiom A1: ≽ is complete and transitive.

VNM Axioms

Axiom A2 (*Independence*): For any p,q,r∈P, and any $a \in (0,1]$,

$$ap + (1-a)r \succ aq + (1-a)r \Leftrightarrow p \succ q.$$



VNM Axioms

Axiom A3 (*Continuity*): For any p,q,r$\in$P, if p $\succ$ q$\succ$ r, then there exist a,$b$$\in$(0,1) such that

ap + (1-a)r $\succ$ q $\succ$ bp + (1-b) r.

Theorem – VNM-representation

A relation $\succcurlyeq$ on P can be represented by a VNM utility function u : Z → R iff $\succcurlyeq$ satisfies Axioms A1-A3.

u and v represent $\succcurlyeq$ iff v = au + b for some a > 0 and b $\in$ R.

Exercise

- Consider a relation $\succcurlyeq$ among positive real numbers represented by VNM utility function u with $u(x) = x^2$.

  Can this relation be represented by VNM utility function $u^*(x) = x^{1/2}$?

  What about $u^{**}(x) = 1/x$?

Normal-form representation

Definition (Normal form): A game is any list

$$G = (S_1, \ldots, S_n; u_1, \ldots, u_n)$$

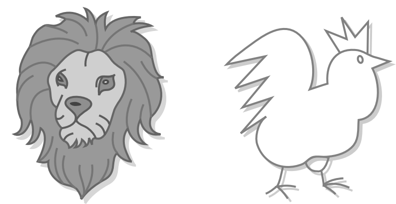
where, for each $i \in N = \{1,2,\ldots,n\}$,

- S_i is the set of all strategies available to i,
- $u_i : S_1 \times \cdots \times S_n \to \Re$ is the VNM utility function of player i.

Assumption: G is common knowledge.

Definition: A player i is rational iff he tries to maximize the expected value of u_i given his beliefs.

Chicken



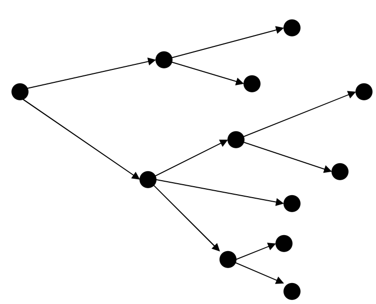
(-1,-1)	(1,0)
(0,1)	(1/2,1/2)

Figures by MIT OCW.

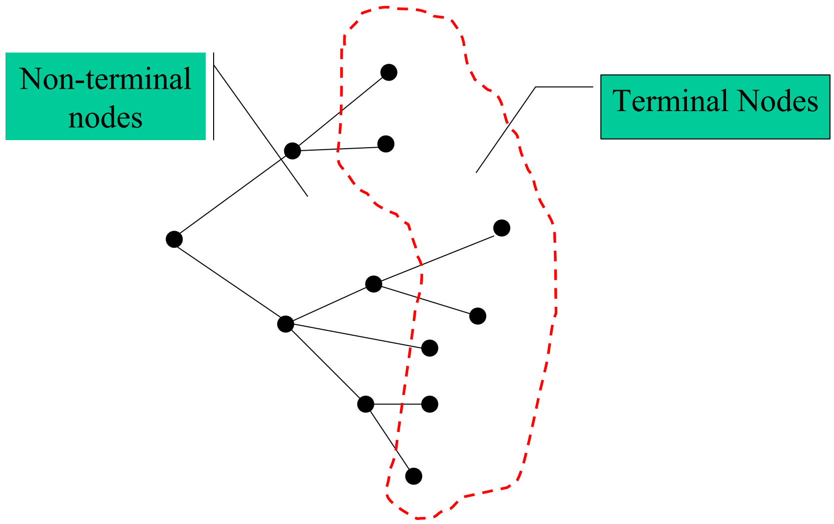
Extensive-form representation

Definition: A **tree** is a set of nodes connected with directed arcs such that

1. There is an initial node;
2. For each other node, there is one incoming arc;
3. each node can be reached through a unique path.



A tree



Extensive form – definition

Definition: A game consists of

– a set of players

– a tree

– an allocation of each non-terminal node to a player

– an informational partition (to be made precise)

– a payoff for each player at each terminal node.

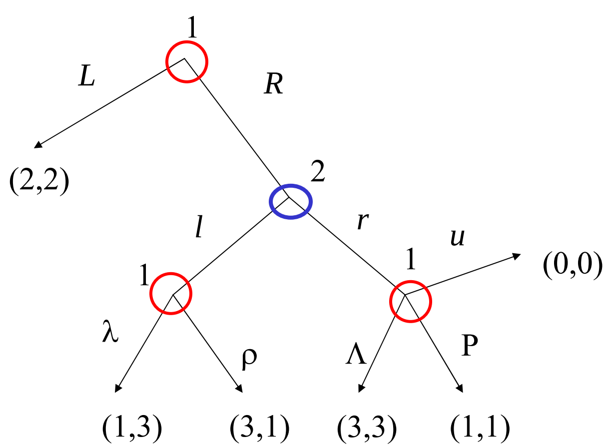
Information set

An **information set** is a collection of nodes such that

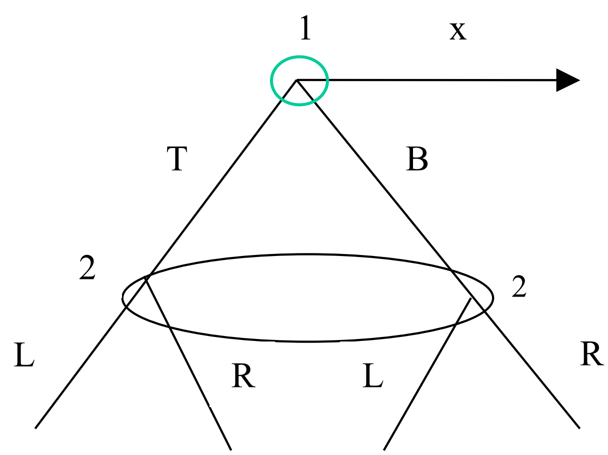
1. The same player is to move at each of these nodes;
2. The same moves are available at each of these nodes.

An **informational partition** is an allocation of each non-terminal node of the tree to an information set.

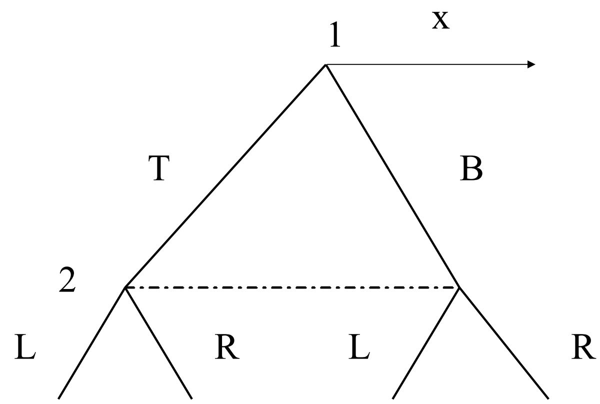
A game



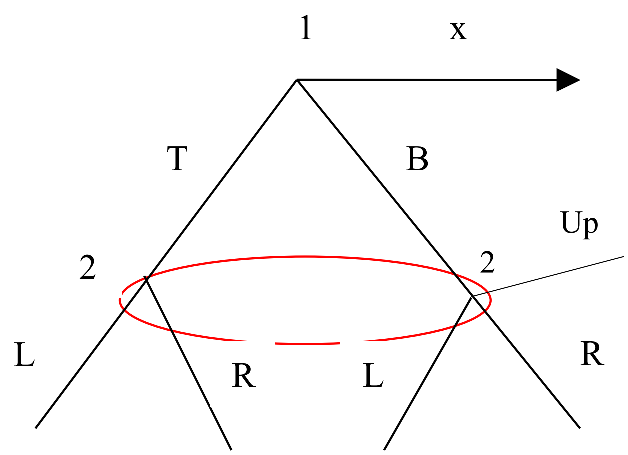
Another game



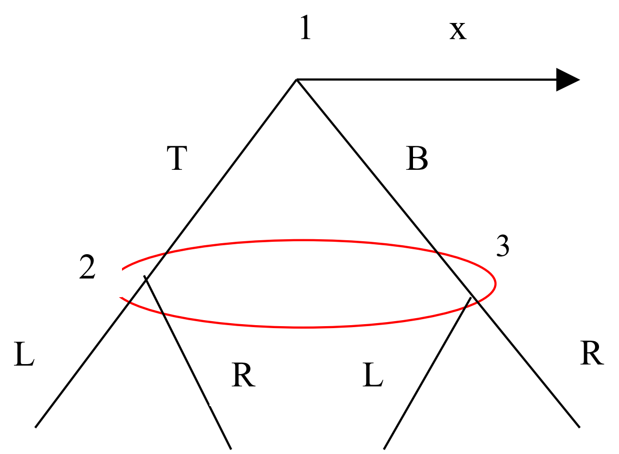
The Same Game



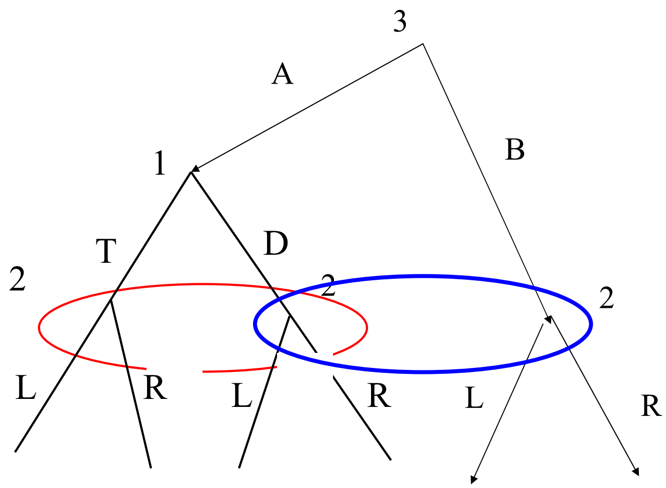
What is wrong?



What is wrong?



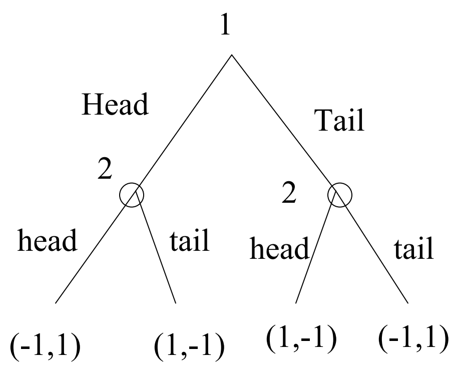
What is wrong?



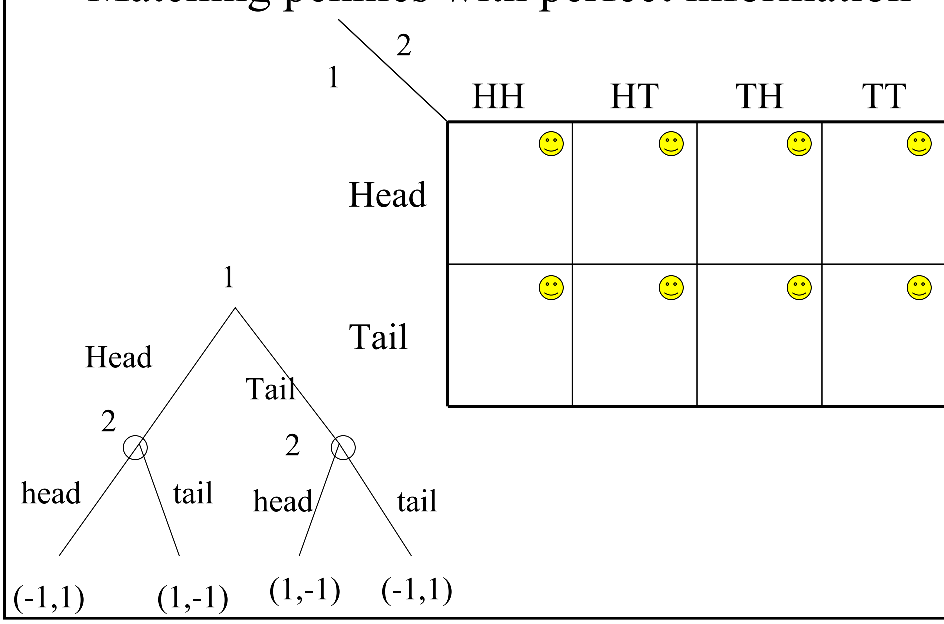
Strategy

A **strategy** of a player **is a complete contingent-plan**, determining which action he will take at each information set he is to move (including the information sets that will not be reached according to this strategy).

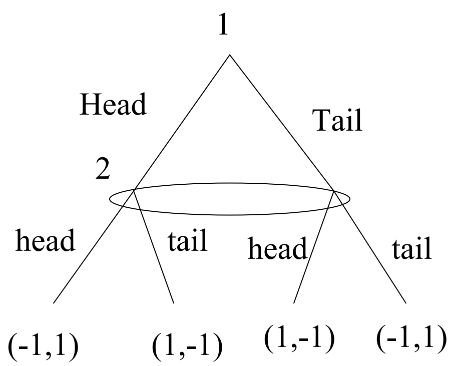
Matching pennies with perfect information



Matching pennies with perfect information

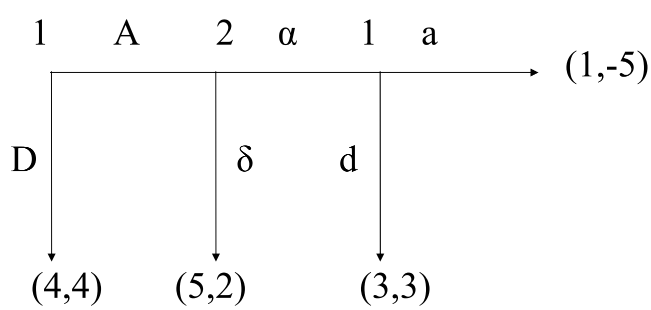


Matching pennies with Imperfect information

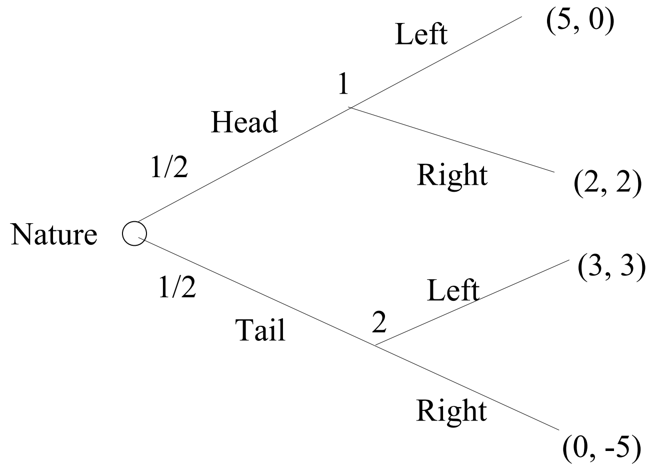


1 \ 2	Head	Tail
Head	(-1,1)	(1,-1)
Tail	(1,-1)	(-1,1)

A game



A game with nature



Mixed Strategy

Definition: A **mixed strategy** of a player is a probability distribution over the set of his strategies.

Pure strategies: $S_i = \{s_{i1}, s_{i2}, \ldots, s_{ik}\}$

A mixed strategy: σ_i: $S \rightarrow [0,1]$ s.t.

$$\sigma_i(s_{i1}) + \sigma_i(s_{i2}) + \ldots + \sigma_i(s_{ik}) = 1.$$

If the other players play $s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$, then the expected utility of playing σ_i is

$$\sigma_i(s_{i1})u_i(s_{i1}, s_{-i}) + \sigma_i(s_{i2})\, u_i(s_{i2}, s_{-i}) + \ldots + \sigma_i(s_{ik})\, u_i(s_{ik}, s_{-i}).$$

How to play

Dominance

$s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$

Definition: A pure strategy s_i^* **strictly dominates** s_i if and only if

$$u_i(s_i^*, s_{-i}) > u_i(s_i, s_{-i}) \quad \forall s_{-i}.$$

A mixed strategy σ_i **strictly dominates** s_i iff

$$\sigma_i(s_{i1})u_i(s_{i1}, s_{-i}) + \cdots + \sigma_i(s_{ik})u_i(s_{ik}, s_{-i}) > u_i(s_i, s_{-i}) \quad \forall s_{-i}$$

A rational player never plays a strictly dominated strategy.

Prisoners' Dilemma

1 \ 2	Cooperate	Defect
Cooperate	(5,5)	(0,6)
Defect	(6,0)	(1,1)

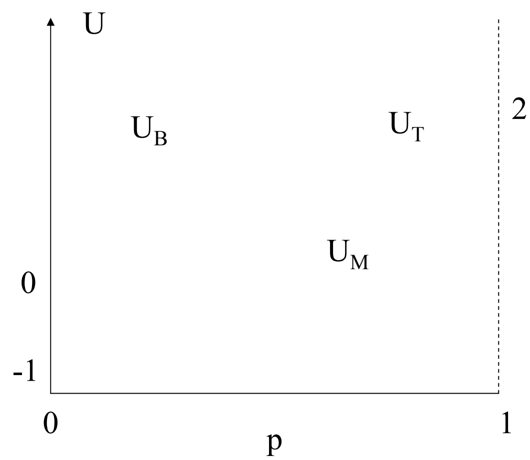
A Game

U_M

$U_T =$

$U_B =$

	L	R
T	(2,0)	(-1,1)
M	(0,10)	(0,0)
B	(-1,-6)	(2,0)
	p	1-p



Weak Dominance

Definition: A pure strategy s_i* weakly **dominates** s_i if and only if

$$u_i(s_i^*, s_{-i}) \geq u_i(s_i , s_{-i}) \quad \forall s_{-i}.$$

and at least one of the inequalities is strict. A mixed strategy σ_i* **weakly dominates** s_i iff

$$\sigma_i(s_{i1})u_i(s_{i1}, s_{-i}) + \cdots + \sigma_i(s_{ik})u_i(s_{ik}, s_{-i}) \geq u_i(s_i, s_{-i}) \quad \forall s_i$$

and at least one of the inequalities is strict.

If a player is rational and cautious (i.e., he assigns positive probability to each of his opponents' strategies), then he will not play a weakly dominated strategy.

Dominant-strategy equilibrium

Definition: A strategy s_i* is a **dominant strategy** iff s_i* **weakly dominates** every other strategy s_i.

Definition: A strategy profile s* is a dominant-strategy equilibrium iff s_i* is a dominant strategy for each player i.

If there is a dominant strategy, then it will be played, so long as the players are …

Prisoners' Dilemma

1 \ 2	Cooperate	Defect
Cooperate	(5,5)	(0,6)
Defect	(6,0)	(1,1)

Second-price auction

- N = {1,2} buyers;
- The value of the house for buyer i is v_i;
- Each buyer i simultaneously bids b_i;
- i* with $b_{i*} = \max b_i$ gets the house and pays the second highest bid

$$p = \max_{j \neq i} b_j.$$

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A Game

	L	R
T	(2,0)	(-1,1)
M	(0,10)	(0,0)
B	(-1,-6)	(2,0)