

6.003: Signals and Systems—Fall 2003

PROBLEM SET 11 SOLUTIONS

Problem 1 (O&W 10.29 (d))

In this problem we are asked to sketch the magnitude of the Fourier transform associated with the pole-zero diagram, Figure P10.29 (d). In order to do so, we need to make some assumptions:

- (a) The ROC includes the unit circle to ensure the existence of the Fourier transform.
- (b) The gain factor is one so that the system we are dealing with has the following form:

$$H(z) = \frac{1}{(z - z_1)(z + z_1)},$$

where z_1 is a positive real number whose magnitude is less than 1.

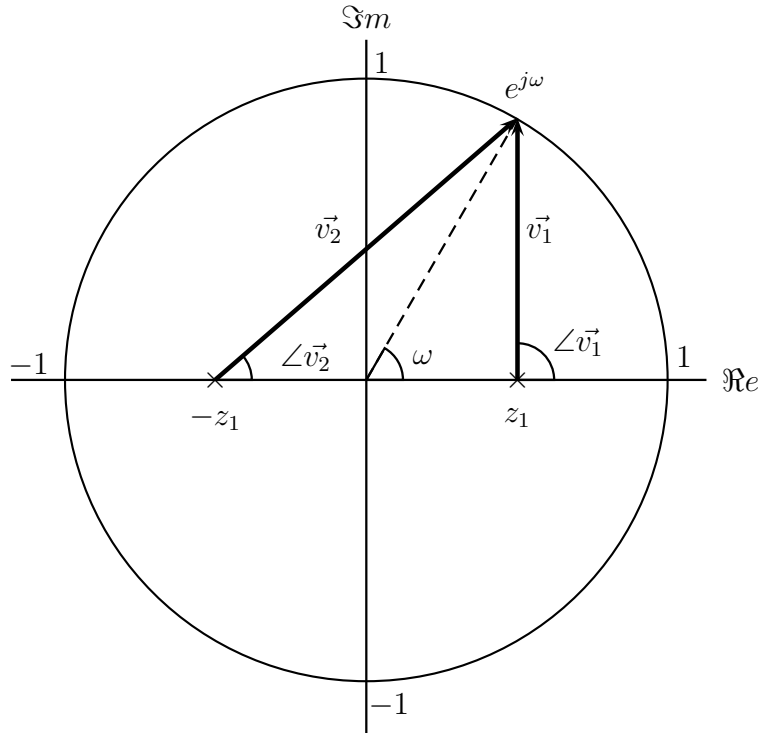
To obtain the frequency response of a DT system, we need to evaluate the magnitude and phase of $H(z)$ along the unit circle in z - plane, *i.e.*, $z = e^{j\omega}$ for $0 \leq \omega < 2\pi$. First, we look at the magnitude plot. In general, for a fixed ω we can think of $|H(e^{j\omega})|$ as

$$|H(e^{j\omega})| = \frac{\prod_{i=1}^{\# \text{ of zeros}} (\text{length of a vector connecting } i \text{ th zero to } e^{j\omega})}{\prod_{j=1}^{\# \text{ of poles}} (\text{length of a vector connecting } j \text{ th pole to } e^{j\omega})}$$

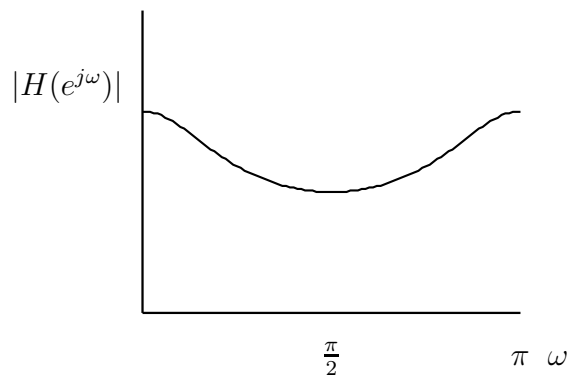
If $\#$ of zeros or poles is zero, then we define the product above to be 1. In our case, there are two poles and no zeros. Thus, the above expression can be simplified to:

$$|H(e^{j\omega})| = \frac{1}{|v_1||v_2|},$$

where v_1 and v_2 are vectors shown in the figure below:



From the plot, we see that when $\omega = \frac{\pi}{2}$, the product of $|v_1|$ and $|v_2|$ becomes maximum. Thus, we expect to have the minimum magnitude at the frequency. Also, when $\omega = 0$ or π , the product of the lengths or the two vectors becomes minimum; thus the magnitude of $H(e^{j\omega})$ becomes maximum. The magnitude plot of $H(e^{j\omega})$ for $0 \leq \omega < \pi$ is shown below:



Although you are not asked to sketch the phase, here is a briefly outline on how to sketch the phase.

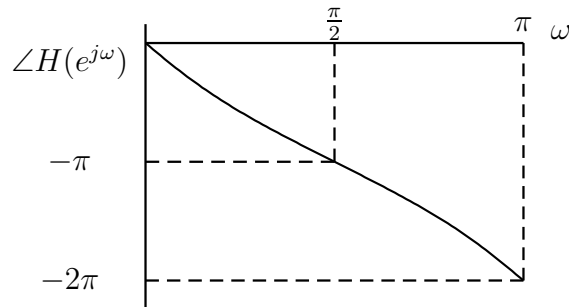
The phase $\angle H(e^{j\omega})$ can be described as:

$$\begin{aligned} \angle H(e^{j\omega}) = & \sum_{i=1}^{\# \text{ of zeros}} (\text{angle of vector connecting } i \text{ th zero to } e^{j\omega}) \\ & - \sum_{j=1}^{\# \text{ of poles}} (\text{angle of vector connecting } j \text{ th pole to } e^{j\omega}). \end{aligned}$$

Again for our specific case, using the vectors v_1 and v_2 defined above we have

$$\angle H(e^{j\omega}) = -(\angle v_1 + \angle v_2),$$

The phase starts off at 0 when $\omega = 0$ and decreases to $-\pi$ when $\omega = \frac{\pi}{2}$. Because of the symmetric poles, the phase keeps decreasing to -2π at $\omega = \pi$. The phase plot is shown below:



Problem 2 (O&W 10.34)

(a)

$$y[n] = y[n - 1] + y[n - 2] + x[n - 1]$$

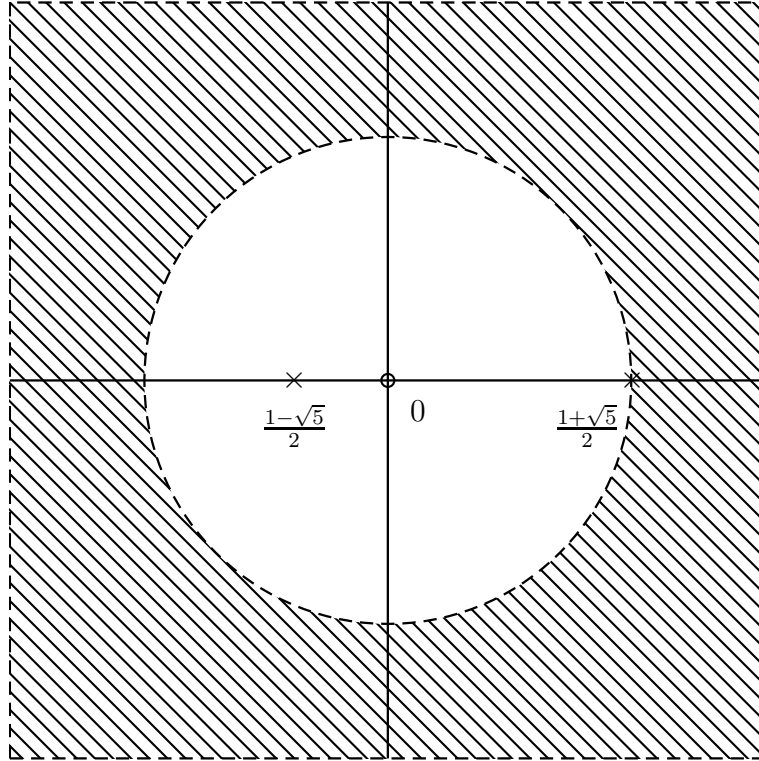
Taking the z-transform of this equation:

$$\begin{aligned} Y(z) &= z^{-1}Y(z) + z^{-2}Y(z) + z^{-1}X(z) \\ H(z) &= \frac{Y(z)}{X(z)} = \frac{z^{-1}}{1 - z^{-1} - z^{-2}} = \frac{z}{z^2 - z - 1} \\ &= \frac{z}{\left(z - \frac{1 + \sqrt{5}}{2}\right) \left(z - \frac{1 - \sqrt{5}}{2}\right)} \end{aligned}$$

$H(z)$ has a zero at $z = 0$ and poles at $z_1 = \frac{1 + \sqrt{5}}{2}$ and $z_2 = \frac{1 - \sqrt{5}}{2}$.

Since the system is causal, the ROC of $H(z)$ will be outside the circle containing its outermost pole: $|z| > |z_1|$. The pzm and ROC are depicted below:

$\Im m$



(b)

$$\begin{aligned} H(z) &= \frac{-z^{-1}}{\left(z^{-1} + \frac{1+\sqrt{5}}{2}\right) \left(z^{-1} + \frac{1-\sqrt{5}}{2}\right)} \\ &= \frac{A}{z^{-1} + \frac{1+\sqrt{5}}{2}} + \frac{B}{z^{-1} + \frac{1-\sqrt{5}}{2}} \\ A &= \left(z^{-1} + \frac{1+\sqrt{5}}{2}\right) H(z) \Big|_{z^{-1} = -\frac{1+\sqrt{5}}{2}} = -\frac{1+\sqrt{5}}{2\sqrt{5}} \\ B &= \left(z^{-1} + \frac{1-\sqrt{5}}{2}\right) H(z) \Big|_{z^{-1} = -\frac{1-\sqrt{5}}{2}} = \frac{1-\sqrt{5}}{2\sqrt{5}} \\ H(z) &= \frac{-\frac{1}{\sqrt{5}}}{1 + \frac{2}{1+\sqrt{5}}z^{-1}} + \frac{\frac{1}{\sqrt{5}}}{1 + \frac{2}{1-\sqrt{5}}z^{-1}}, \quad |z| > \frac{1+\sqrt{5}}{2} \end{aligned}$$

Taking the inverse z-transform:

$$h[n] = -\frac{1}{\sqrt{5}} \left(-\frac{2}{1+\sqrt{5}} \right)^n u[n] + \frac{1}{\sqrt{5}} \left(-\frac{2}{1-\sqrt{5}} \right)^n u[n]$$

- (c) The system is unstable, as its ROC does not contain the unit circle. The instability is also apparent in $h[n]$, as the $\left(-\frac{2}{1-\sqrt{5}} \right)^n$ term will grow indefinitely as $n \rightarrow \infty$.

In order to make the system stable, the ROC must contain the unit circle. For stability, the ROC should be: $\frac{2}{-1-\sqrt{5}} < |z| < \frac{2}{-1+\sqrt{5}}$.

The inverse z-transform of $H(z)$ with this ROC is:

$$h[n] = -\frac{1}{\sqrt{5}} \left(-\frac{2}{1+\sqrt{5}} \right)^n u[n] - \frac{1}{\sqrt{5}} \left(-\frac{2}{1-\sqrt{5}} \right)^n u[-n-1]$$

Problem 3 (O&W 10.42)

Because we are dealing with a system which has initial conditions, we may want to use the unilateral z-transform. From the properties of the unilateral z-transform, we get the following relationships:

$$\begin{aligned} y[n] &\longleftrightarrow Y(z) \\ y[n-1] &\longleftrightarrow z^{-1}Y(z) + y[-1] \end{aligned}$$

In each part of this problem, the first step is to take the unilateral z-transform of both sides of the difference equation. To find the zero-input response (ZIR), set the input to 0 and solve for $Y(z)$. To find the zero-state response (ZSR), set the initial conditions to zero and solve for $Y(z)$.

- (a)

$$y[n] + 3y[n-1] = x[n], \quad y[-1] = 1, \quad x[n] = \left(\frac{1}{2} \right)^n u[n]$$

Taking the unilateral z-transform of both sides of the difference equation,

$$Y(z) + 3z^{-1}Y(z) + 3y[-1] = X(z).$$

Solving for $Y(z)$ gives,

$$Y(z) = \underbrace{\frac{X(z)}{1+3z^{-1}}}_{ZSR} + \underbrace{\frac{-3y[-1]}{1+3z^{-1}}}_{ZIR}.$$

To find the ZIR, set $X(z) = 0$ and use the fact that $y[-1] = 1$,

$$\begin{aligned} Y_{ZIR}(z) &= \frac{-3}{1 + 3z^{-1}} \\ y_{ZIR}[n] &= -3(-3)^n u[n] = (-3)^{n+1} u[n]. \end{aligned}$$

To find the ZSR, set $y[-1] = 0$ and use $x[n] = \left(\frac{1}{2}\right)^n u[n]$ as given in the problem,

$$\begin{aligned} X(z) &= \frac{1}{1 - \frac{1}{2}z^{-1}} \\ Y_{ZSR}(z) &= \frac{1}{(1 + 3z^{-1})(1 - \frac{1}{2}z^{-1})} = \frac{\frac{6}{7}}{1 + 3z^{-1}} + \frac{\frac{1}{7}}{1 - \frac{1}{2}z^{-1}} \\ y_{ZSR}[n] &= \frac{6}{7}(-3)^n u[n] + \frac{1}{7} \left(\frac{1}{2}\right)^n u[n]. \end{aligned}$$

(b)

$$y[n] - \frac{1}{2}y[n-1] = x[n] - \frac{1}{2}x[n-1], \quad y[-1] = 0, \quad x[n] = u[n]$$

Taking the unilateral z -transform of both sides of the difference equation,

$$Y(z) - \frac{1}{2}z^{-1}Y(z) - \frac{1}{2}y[-1] = X(z) - \frac{1}{2}z^{-1}X(z) - \frac{1}{2}x[-1].$$

Solving for $Y(z)$ gives,

$$\begin{aligned} Y(z) &= \frac{(1 - \frac{1}{2}z^{-1})X(z)}{1 - \frac{1}{2}z^{-1}} + \frac{-\frac{1}{2}x[-1] + \frac{1}{2}y[-1]}{1 - \frac{1}{2}z^{-1}} \\ &= \underbrace{X(z)}_{ZSR} + \underbrace{\frac{-\frac{1}{2}x[-1] + \frac{1}{2}y[-1]}{1 - \frac{1}{2}z^{-1}}}_{ZIR}. \end{aligned}$$

To find the ZIR, set $X(z) = 0$ (note that this means $x[-1] = 0$), and use the fact that $y[-1] = 0$,

$$Y_{ZIR}(z) = 0 \implies y_{ZIR}[n] = 0.$$

To find the ZSR, set $y[-1] = 0$ and use $x[n] = u[n]$ as given in the problem,

$$\begin{aligned} X(z) &= \frac{1}{1 - z^{-1}} \\ Y_{ZSR}(z) &= \frac{1}{1 - z^{-1}} \\ y_{ZSR}[n] &= u[n]. \end{aligned}$$

(c)

$$y[n] - \frac{1}{2}y[n-1] = x[n] - \frac{1}{2}x[n-1], \quad y[-1] = 1, \quad x[n] = u[n]$$

Since this is the same difference equation as given in part (b), we can use the equation for $Y(z)$ derived above. To find the ZIR, set $X(z) = 0$ (note that this means $x[-1] = 0$), and use the fact that $y[-1] = 1$,

$$\begin{aligned} Y_{ZIR}(z) &= \frac{\frac{1}{2}}{1 - \frac{1}{2}z^{-1}} \\ y_{ZIR}[n] &= \frac{1}{2} \left(\frac{1}{2}\right)^n u[n] = \left(\frac{1}{2}\right)^{n+1} u[n]. \end{aligned}$$

To find the ZSR, set $y[-1] = 0$ and use $x[n] = u[n]$ as given in the problem. This is exactly what we did in part (b) above, so

$$y_{ZSR}[n] = u[n].$$

Problem 4 (O&W 10.47)

(a) Recall that complex exponentials are the eigenfunction of LTI systems. Thus, the response of LTI systems to complex exponentials is of the form:

$$z^n \longrightarrow H(z)z^n.$$

Since the output of the system to an input $x[n] = (-2)^n$ is 0, we conclude that the ROC of $H(z)$ contains $z = -2$ and $H(z)$ has a zero at $z = -2$.

The second input-output relation gives us:

$$\begin{aligned} H(z) &= \frac{Y(z)}{X(z)} = \frac{1 + \frac{a}{1 - \frac{1}{4}z^{-1}}}{\frac{1}{1 - \frac{1}{2}z^{-1}}}, \quad |z| > \frac{1}{2} \\ &= \frac{(1 + a - \frac{1}{4}z^{-1})(1 - \frac{1}{2}z^{-1})}{1 - \frac{1}{4}z^{-1}} \\ &= (1 + a) \frac{(z - \frac{1}{4(1+a)})(z - \frac{1}{2})}{z(z - \frac{1}{4})}, \end{aligned}$$

assuming that $1 + a \neq 0$. $H(z)$ has two poles; at $z = 0$ and $z = \frac{1}{4}$. In order to include $z = -2$ in its ROC, we know that the system is causal and its ROC is outside of the circle of radius $\frac{1}{4}$. Since $H(z)$ has a zero at $z = -2$,

$$1 + a - \frac{1}{4}z^{-1}|_{z=-2} = 0 \Rightarrow a = -\frac{9}{8}.$$

(b) Now we know the expression for $H(z)$:

$$H(z) = \frac{\left(-\frac{1}{8} - \frac{1}{4}z^{-1}\right)\left(1 - \frac{1}{2}z^{-1}\right)}{1 - \frac{1}{4}z^{-1}}.$$

The input in this case is a complex exponential $x[n] = 1^n$.
The output will therefore be of the form:

$$\begin{aligned} y[n] &= H(1) \cdot 1^n = \frac{\left(-\frac{3}{8}\right)\left(\frac{1}{2}\right)}{\frac{3}{4}} \\ &= -\frac{1}{4}. \end{aligned}$$

Problem 5 (O&W 10.50)

(a) From the pole-zero pattern, the system function takes the following form:

$$H(z) = \frac{z - \frac{1}{a}}{z - a}.$$

Showing that $|H(e^{j\omega})|$ is constant is equivalent to showing that $|H(e^{j\omega})|^2$ is constant.

$$\begin{aligned} |H(e^{j\omega})|^2 &= H(e^{j\omega})H^*(e^{j\omega}) = \frac{e^{j\omega} - \frac{1}{a}}{e^{j\omega} - a} \cdot \frac{e^{-j\omega} - \frac{1}{a}}{e^{-j\omega} - a} \\ &= \frac{1 - \frac{2}{a} \cos(\omega) + \frac{1}{a^2}}{1 - 2a \cos(\omega) + a^2} = \frac{1}{a^2} \cdot \frac{a^2 - 2a \cos(\omega) + 1}{1 - 2a \cos(\omega) + a^2} \\ &= \frac{1}{a^2}. \end{aligned}$$

Thus, $|H(e^{j\omega})| = \frac{1}{|a|}$ which is constant and is completely determined by the location of the pole and zero.

(b) Using the law of cosines,

$$\begin{aligned} |v_1|^2 &= |1|^2 + |a|^2 - 2|1||a| \cos(\omega) \\ &= 1 + a^2 - 2a \cos(\omega). \end{aligned}$$

(c) Following the same procedure as in (b),

$$\begin{aligned} |v_2|^2 &= |1|^2 + \left|\frac{1}{a}\right|^2 - 2|1|\left|\frac{1}{a}\right| \cos(\omega) \\ &= \frac{1}{a^2} [a^2 + 1 - 2a \cos(\omega)] = \frac{1}{a^2} |v_1|^2. \\ \therefore |v_2| &= \left|\frac{v_1}{a}\right|. \end{aligned}$$

It clearly shows that the length of v_2 is proportional in length to v_1 independently of ω .

Problem 6 (O&W 11.25 (a))

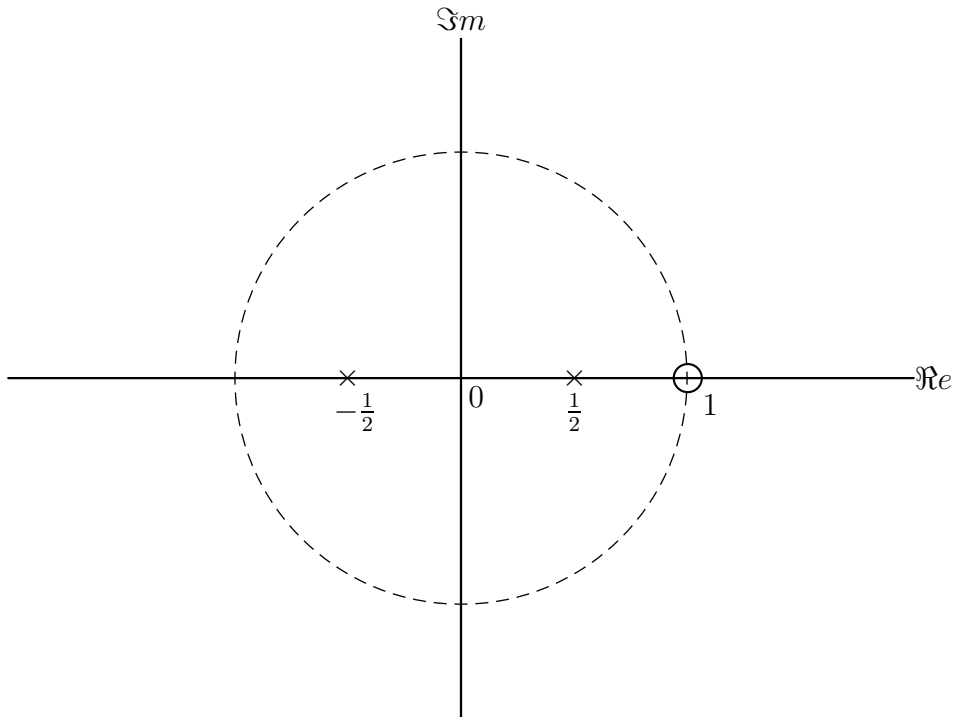
The system given is:

$$G(z)H(z) = \frac{z - 1}{z^2 - \frac{1}{4}}.$$

First, we would like to find the poles and zeros of the system. It is easy to see that $G(z)H(z)$ can be expressed as:

$$G(z)H(z) = \frac{z - 1}{(z + \frac{1}{2})(z - \frac{1}{2})}. \quad (1)$$

The system has poles at $\pm\frac{1}{2}$ and a zero at 1. The pole-zero map as is as shown below. Note that dashed circle in all the figures in this problem denotes a unit circle.



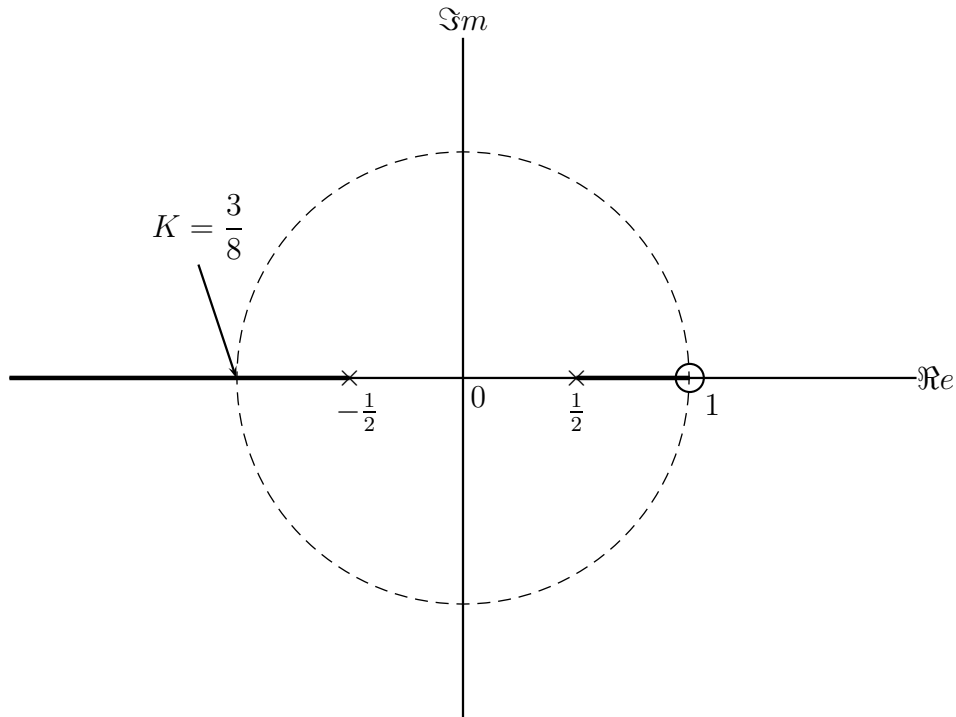
Recall that the derivation of the angle criteria does not concern if the system is CT or DT. Thus, by evoking the angle criteria we have:

$$\begin{aligned} \angle G(z)H(z) &= \text{odd integer multiple of } \pi \text{ if } K > 0, \\ \angle G(z)H(z) &= \text{even integer multiple of } \pi \text{ if } K < 0. \end{aligned}$$

for all z on the locus.

Let's look at the case where $K > 0$. Using the angle criteria, we can identify that the two real line segments belong to the locus. One is $(-\infty, -\frac{1}{2})$ and the other $(\frac{1}{2}, 1)$. In this case, these two segments completely specify the locus. From this, we can see that a pole at $\frac{1}{2}$ will approach to the zero at 1 as $K \rightarrow \infty$. The other pole, one at $-\frac{1}{2}$ will move to $-\infty$. Thus, as $K \rightarrow \infty$, one of the poles will become unstable (Note that in feedback systems, we only consider causal systems). Thus, we would like to know up to which K , the closed loop system remains stable, i.e., we would like to find the value of K when the closed loop system has a pole at -1 . From Eqn (1),

$$\begin{aligned} G(z)H(z)|_{z=-1} &= -\frac{1}{K} \\ \frac{z-1}{(z+\frac{1}{2})(z-\frac{1}{2})}\Big|_{z=-1} &= -\frac{1}{K} \\ \frac{-1-1}{(-1+\frac{1}{2})(-1-\frac{1}{2})} &= -\frac{1}{K} \\ \therefore K &= \frac{3}{8}. \end{aligned}$$



Thus, the root locus for $K > 0$ is shown above.

Now let's look at the case where $K < 0$. If we apply the angle criteria only on the real axis, then we identify that again there are two segments belonging to the locus. One is $(-\frac{1}{2}, \frac{1}{2})$ and the other is $(1, \infty)$. As we know, all the poles in $G(z)H(z)$ will go to the zeros in $G(z)H(z)$ as $|K| \rightarrow \infty$. Thus, at some point on $(-\frac{1}{2}, \frac{1}{2})$, the locus branches off from the real axis and go to a point on $(1, \infty)$ to reach to the zero at 1 and at ∞ . Thus, first we would like to know where those branching, or usually referred to bifurcation, points are. At bifurcation points, multiple poles exist at those points; in our case there will be two identical poles at bifurcation points. Since there are only two poles for the system, we want to find a value of K such that

$$1 + KG(z)H(z) = 0 \quad (2)$$

has double-root.

$$\begin{aligned} 1 + K \frac{z-1}{z^2 - \frac{1}{4}} &= 0 \\ z^2 - \frac{1}{4} + K(z-1) &= 0 \\ \left(z + \frac{K}{2}\right)^2 - \left(\frac{K}{2}\right)^2 - K - \frac{1}{4} &= 0 \quad \text{by completing the square.} \end{aligned}$$

Thus, in order to have multiple roots K needs to satisfy the following:

$$\begin{aligned} \left(\frac{K}{2}\right)^2 + K + \frac{1}{4} &= 0 \\ \therefore K &= -2 \pm \sqrt{3}. \end{aligned}$$

As expected, both K values are indeed negative, and at those values of K , bifurcation takes place and the corresponding locations are at $z = -\frac{K}{2} = 1 \mp \frac{\sqrt{3}}{2}$.

Can we find the value of K up to which the closed loop system remains stable as K changes from 0 to $-\infty$? In this case, it is not as easy as in the case of $K > 0$ since we do not know where the locus crosses the unit circle. However, we do know that on the unit circle, $z = e^{j\omega_0}$ for some ω_0 . Thus, from Eqn (2), we have:

$$\begin{aligned} G(z)H(z)|_{z=e^{j\omega_0}} &= -\frac{1}{K_0} \\ \frac{e^{j\omega_0} - 1}{e^{2j\omega_0} - \frac{1}{4}} &= -\frac{1}{K_0}, \end{aligned} \quad (3)$$

where K_0 corresponds to ω_0 .

One thing to note is that $G(z)H(z)$ is a rational function in z and all the coefficients are real. Thus, the locus is symmetric about the real axis; specifically the points where the locus

crosses the unit circle are of complex conjugate. Thus, with the same K_0 as in Eqn (3) the following holds:

$$\begin{aligned} G(z)H(z)|_{z=e^{-j\omega_0}} &= -\frac{1}{K_0} \\ \frac{e^{-j\omega_0} - 1}{e^{-2j\omega_0} - \frac{1}{4}} &= -\frac{1}{K_0}, \end{aligned} \quad (4)$$

and combining Eqns (3) and (4), we get:

$$\begin{aligned} \frac{e^{-j\omega_0} - 1}{e^{-2j\omega_0} - \frac{1}{4}} &= \frac{e^{j\omega_0} - 1}{e^{2j\omega_0} - \frac{1}{4}} \\ (e^{-j\omega_0} - 1) \left(e^{2j\omega_0} - \frac{1}{4} \right) &= (e^{j\omega_0} - 1) \left(e^{-2j\omega_0} - \frac{1}{4} \right) \\ e^{j\omega_0} - e^{2j\omega_0} - \frac{1}{4}e^{-j\omega_0} + \frac{1}{4} &= e^{-j\omega_0} - e^{-2j\omega_0} - \frac{1}{4}e^{j\omega_0} + \frac{1}{4} \\ \frac{5}{4}(e^{j\omega_0} - e^{-j\omega_0}) &= (e^{2j\omega_0} - e^{-2j\omega_0}) \\ \frac{\sin \omega_0}{\sin 2\omega_0} = \frac{\sin \omega_0}{2 \sin \omega_0 \cos \omega_0} &= \frac{4}{5} \\ \therefore \cos \omega_0 &= \frac{5}{8}. \end{aligned}$$

This result along with Eqn(3) yields

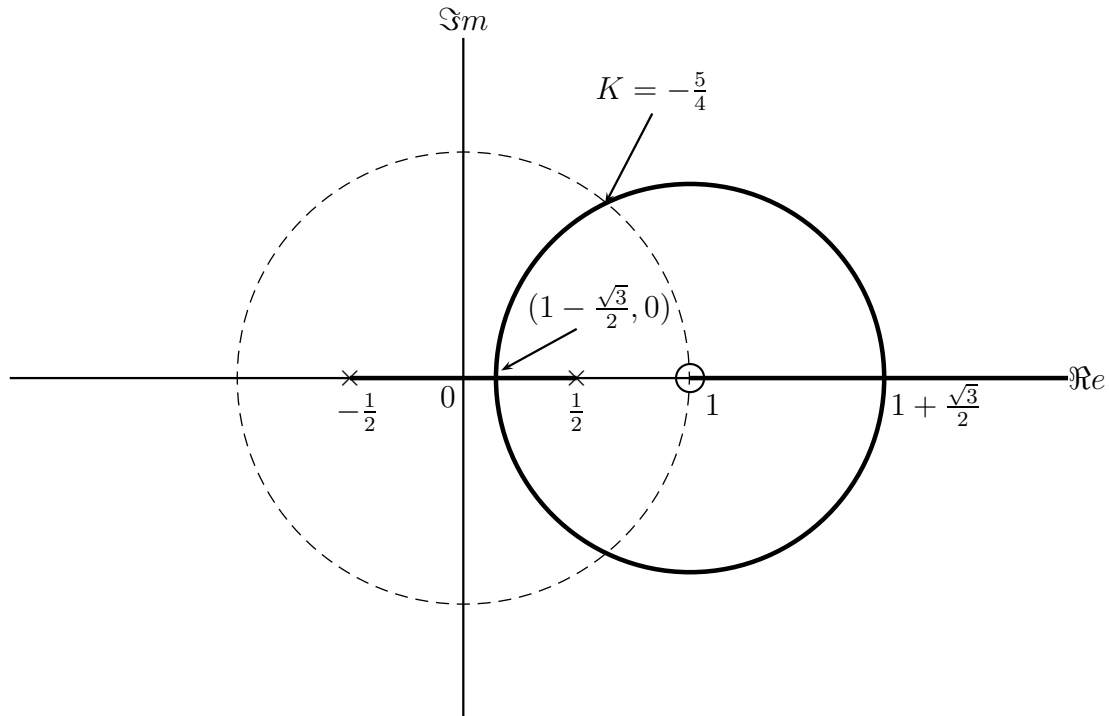
$$K_0 = -\frac{5}{4}.$$

Finally, we want to determine the expression of the locus between the bifurcation points. Let $z = \sigma + j\omega$ and find the relation between σ and ω . From Eqn (2):

$$\begin{aligned} 1 + K \frac{z-1}{z^2 - \frac{1}{4}} &= 0 \\ z^2 - \frac{1}{4} + K(z-1) &= 0 \\ (\sigma + j\omega)^2 + K(\sigma + j\omega) - \frac{1}{4} - K &= 0 \\ \text{For real part: } \sigma^2 - \omega^2 + K\sigma - \frac{1}{4} - K &= 0 \\ \text{For imaginary part: } 2\sigma\omega + K\omega &= 0 \rightarrow K = -2\sigma \\ \sigma^2 - \omega^2 - 2\sigma^2 - \frac{1}{4} + 2\sigma &= 0 \\ \therefore (\sigma - 1)^2 + \omega^2 &= \left(\frac{\sqrt{3}}{2} \right)^2. \end{aligned}$$

The locus traces a circle whose center is at $(\sigma, \omega) = (1, 0)$ in the complex plane and whose radius is $\frac{\sqrt{3}}{2}$.

Thus, the complete root locus for $K < 0$ case is shown below:



Problem 7 (O&W 11.59)

(a) Use Black's equation to compute the transfer function from $x[n]$ to $e[n]$ for the system:

$$\frac{Y(z)}{X(z)} = \frac{H(z)}{1 + H(z)} = \frac{1}{(z-1)(z+\frac{1}{2}) + 1} = \frac{1}{z^2 - \frac{1}{2}z + \frac{1}{2}}$$

$$\frac{E(z)}{X(z)} = 1 - \frac{Y(z)}{X(z)} = \frac{z^2 - \frac{1}{2}z - \frac{1}{2}}{z^2 - \frac{1}{2}z + \frac{1}{2}}$$

The transfer function has two poles at $z = \frac{1}{4} \pm j\sqrt{\frac{7}{16}}$. Note that the magnitude of each pole is $1/\sqrt{2} \approx 0.707$, so the system is stable.

Since the system is stable and LTI, it has a frequency response, and we can use a familiar trick. To find the steady-state response to a step input, evaluate the frequency response

at $\omega = 0$ ($z = e^{j\omega} = 1$):

$$x[n] = u[n] \implies \lim_{n \rightarrow \infty} e[n] = \left. \frac{z^2 - \frac{1}{2}z - \frac{1}{2}}{z^2 - \frac{1}{2}z + \frac{1}{2}} \right|_{z=e^{j0}=1} = 0$$

(b) Note that our equation for the E/X transfer function can be rewritten as follows:

$$\frac{E(z)}{X(z)} = 1 - \frac{Y(z)}{X(z)} = 1 - \frac{H(z)}{1 + H(z)} = \frac{1}{1 + H(z)}$$

If $x[n] = u[n]$, then $X(z) = \frac{1}{1-z^{-1}}$, and we have the following:

$$E(z) = \frac{z}{z-1} \cdot \frac{1}{1+H(z)}$$

We assume the system is stable, so all the roots of $1 + H(z)$ must lie within the unit circle. In particular, given the bounded input $x[n] = u[n]$, we know that the output $e[n]$ must be bounded; all the poles of $E(z)$ must lie within the unit circle. Since the necessary Fourier transforms exist, we can use the familiar trick:

$$x[n] = u[n] \implies \lim_{n \rightarrow \infty} e[n] = \left. \frac{1}{1 + H(z)} \right|_{z=e^{j0}=1} = 0$$

The last step follows because we assume that $H(z)$ has a pole at $z = 1$. Thus the *denominator* becomes unbounded as $z \rightarrow 1$, and the system response goes to zero.

(c) Plug the new transfer function into our equations:

$$\frac{E(z)}{X(z)} = \frac{1}{1 + H(z)} = \frac{1 - z^{-1}}{1 - z^{-1} + z^{-1}} = 1 - z^{-1}$$

This system has one pole at $z = 0$; it is stable. By inspection, the impulse response of the error is $e[n] = \delta[n] - \delta[n - 1]$, and the step response of the error is $e[n] = \delta[n]$; the system tracks a unit step input without error after one time-step.

(d) Plug the new transfer function into our equations:

$$\begin{aligned} \frac{E(z)}{X(z)} &= \frac{1}{1 + H(z)} \\ &= \frac{(1 + \frac{1}{4}z^{-1})(1 - z^{-1})}{(1 + \frac{1}{4}z^{-1})(1 - z^{-1}) + \frac{3}{4}z^{-1} + \frac{1}{4}z^{-2}} \\ &= \frac{1 - \frac{3}{4}z^{-1} - \frac{1}{4}z^{-2}}{1 - \frac{3}{4}z^{-1} - \frac{1}{4}z^{-2} + \frac{3}{4}z^{-1} + \frac{1}{4}z^{-2}} = 1 - \frac{3}{4}z^{-1} - \frac{1}{4}z^{-2} \end{aligned}$$

This system has two poles at $z = 0$; it is stable. By inspection, the impulse response of the error is $e[n] = \delta[n] - \frac{3}{4}\delta[n - 1] - \frac{1}{4}\delta[n - 2]$, and the step response of the error is $e[n] = \delta[n] + \frac{1}{4}\delta[n - 1]$; the system tracks a unit step input without error after two time-steps.

(e) We are given the desired step response of the error signal $e[n]$:

$$e[n] = \sum_{k=0}^{N-1} a_k \delta[n-k]$$

$$\implies E(z) = \sum_{k=0}^{N-1} a_k z^{-k}$$

From our work in part (b), we also have the following equation for the step response of $e[n]$:

$$E(z) = \frac{1}{1-z^{-1}} \cdot \frac{1}{1+H(z)}$$

Set the two equations equal and solve for $H(z)$:

$$\sum_{k=0}^{N-1} a_k z^{-k} = \frac{1}{1-z^{-1}} \cdot \frac{1}{1+H(z)}$$

$$1+H(z) = \frac{1}{1-z^{-1}} \cdot \frac{1}{\sum_{k=0}^{N-1} a_k z^{-k}}$$

$$H(z) = \frac{1 - (1-z^{-1}) \sum_{k=0}^{N-1} a_k z^{-k}}{(1-z^{-1}) \sum_{k=0}^{N-1} a_k z^{-k}}$$

(f) Find the transform of $x[n]$ using the tables:

$$x[n] = (n+1)u[n] = nu[n] + u[n]$$

$$\implies X(z) = \frac{z^{-1}}{(1-z^{-1})^2} + \frac{1}{1-z^{-1}} = \frac{1}{(1-z^{-1})^2}$$

Plug the new transfer function $H(z)$ into our equations:

$$E(z) = X(z) \frac{1}{1+H(z)} = \frac{1}{(1-z^{-1})^2} \cdot \frac{(1+z^{-1})(1-z^{-1})^2}{(1+z^{-1})(1-z^{-1})^2 + z^{-1} + z^{-2} - z^{-3}}$$

$$= \frac{1+z^{-1}}{(1+z^{-1})(1-2z^{-1}+z^{-2}) + z^{-1} + z^{-2} - z^{-3}}$$

$$= 1+z^{-1}$$

By inspection, $e[n] = \delta[n] + \delta[n-1]$, so the system tracks perfectly after two time-steps.