Signals and Systems

Fall 2003

Lecture #22

2 December 2003

1. Properties of the ROC of the z-Transform
2. Inverse z-Transform
3. Examples
4. Properties of the z-Transform
5. System Functions of DT LTI Systems
   a. Causality
   b. Stability

The z-Transform

$$x[n] \longleftrightarrow X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \mathcal{Z}\{x[n]\}$$

$$\text{ROC} = \left\{ z = re^{j\omega} \text{ at which } \sum_{n=-\infty}^{\infty} |x[n] r^{-n}| < \infty \right\}$$

-depends only on $r = |z|$, just like the ROC in s-plane only depends on $Re(s)$

- Last time:
  - Unit circle ($r = 1$) in the ROC $\Rightarrow$ DTFT $X(e^{j\omega})$ exists
  - Rational transforms correspond to signals that are linear combinations of DT exponentials

Some Intuition on the Relation between *z*T and LT

$$x(t) \longleftrightarrow X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt = \mathcal{L}\{x(t)\}$$

Let $t = nT$

$$= \lim_{T \to 0} \sum_{n=-\infty}^{\infty} \underbrace{x(nT)}_{x[n]}(e^{sT})^{-n} \cdot T$$

$$= \lim_{T \to 0} T \sum_{n=-\infty}^{\infty} x[n](e^{sT})^{-n}$$

The (Bilateral) z-Transform

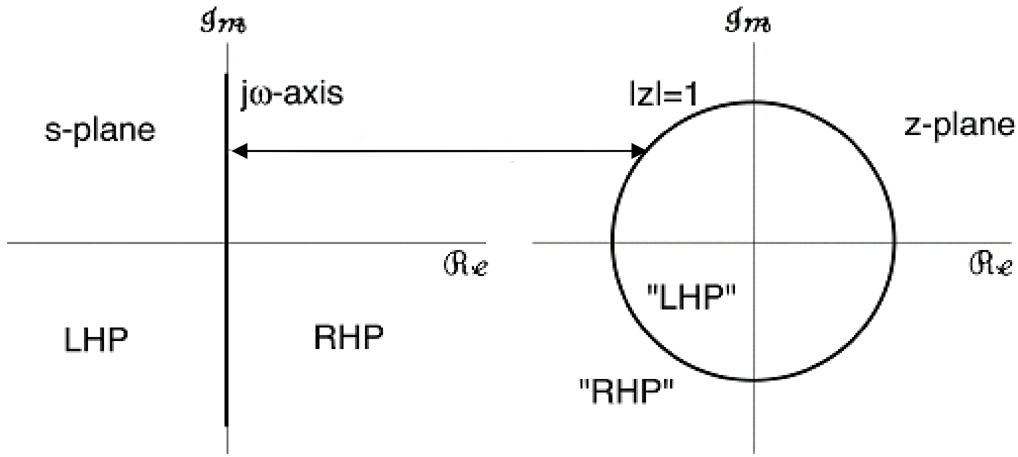
$$x[n] \longleftrightarrow X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \mathcal{Z}\{x[n]\}$$

Can think of z-transform as DT version of Laplace transform with $z = e^{sT}$

More intuition on *z*T-*L*T, *s*-plane - *z*-plane relationship

$$e^{sT} = z$$

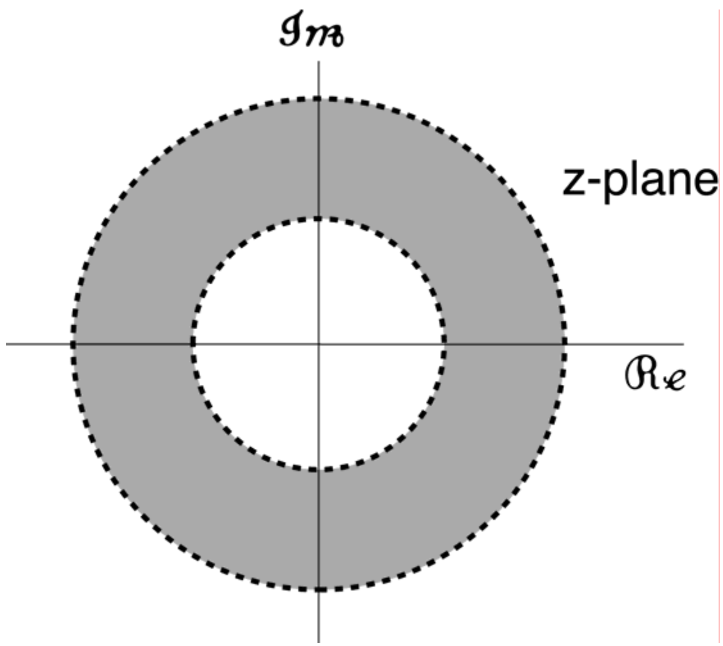
$j\omega$ axis in s-plane ($s = j\omega$) $\Leftrightarrow$ $|z| = |e^{j\omega T}| = 1$ - a unit circle in z-plane



- LHP in s-plane, $Re(s) < 0 \Rightarrow |z| = |e^{sT}| < 1$, inside the $|z| = 1$ circle.
  Special case, $Re(s) = -\infty \Leftrightarrow |z| = 0$.
- RHP in s-plane, $Re(s) > 0 \Rightarrow |z| = |e^{sT}| > 1$, outside the $|z| = 1$ circle.
  Special case, $Re(s) = +\infty \Leftrightarrow |z| = \infty$.
- A vertical line in s-plane, $Re(s)$ = constant $\Leftrightarrow$ $|e^{sT}|$ = constant, a circle in z-plane.

Properties of the ROCs of *z*-Transforms

(1) The ROC of $X(z)$ consists of a ring in the z-plane centered about the origin (equivalent to a vertical strip in the s-plane)



(2) The ROC does *not* contain any poles (same as in *LT*).

More ROC Properties

(3) If $x[n]$ is of finite duration, then the ROC is the entire z-plane, except possibly at $z = 0$ and/or $z = \infty$.

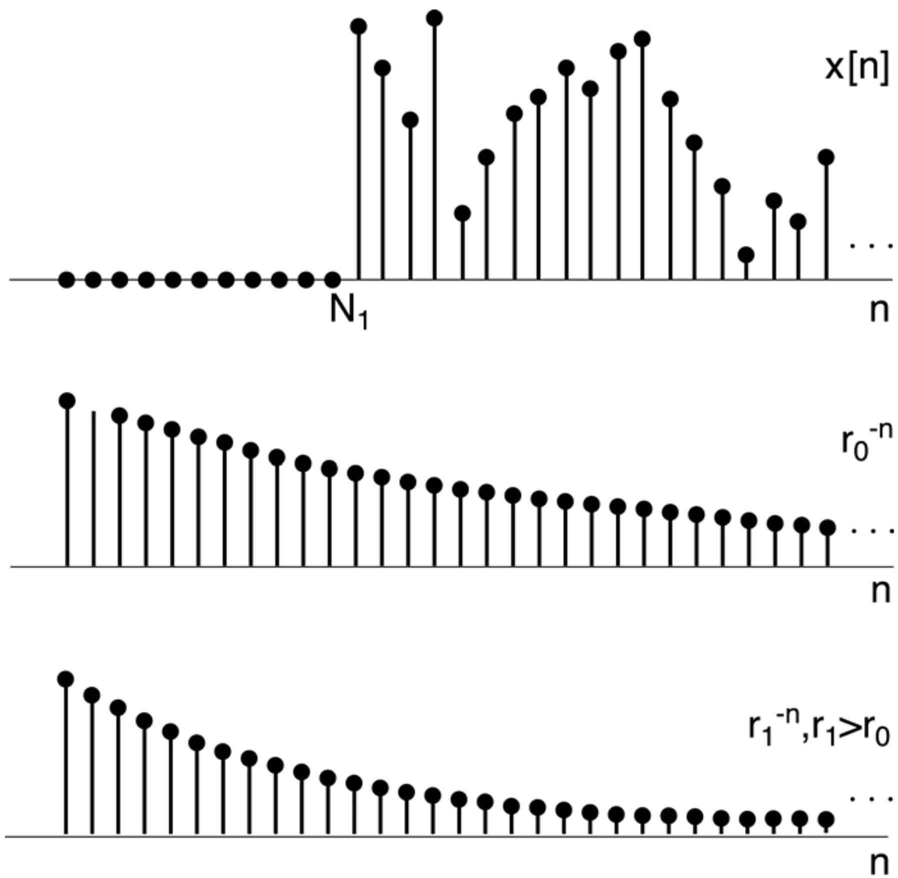
Why?

$$X(z) = \sum_{n=N_1}^{N_2} x[n] z^{-n}$$

Examples:				CT counterpart			
$\delta[n]$	$\longleftrightarrow$	1	ROC all z	$\delta(t)$	$\longleftrightarrow$	1	ROC all s
$\delta[n-1]$	$\longleftrightarrow$	z^{-1}	ROC $z \neq 0$	$\delta(t-T)$	$\longleftrightarrow$	e^{-sT}	$\Re e\{s\} \neq -\infty$
$\delta[n+1]$	$\longleftrightarrow$	z	ROC $z \neq \infty$	$\delta(t+T)$	$\longleftrightarrow$	e^{sT}	$\Re e\{s\} \neq \infty$

ROC Properties Continued

(4) If $x[n]$ is a right-sided sequence, and if $|z| = r_o$ is in the ROC, then all finite values of z for which $|z| > r_o$ are also in the ROC.



$$\sum_{n=N_1}^{\infty} x[n] r_1^{-n}$$

converges faster than

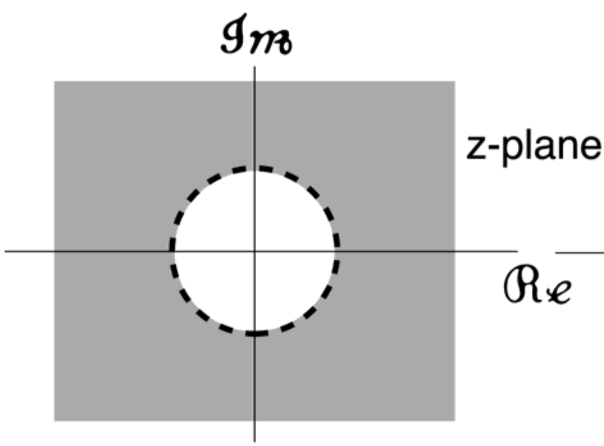
$$\sum_{n=N_1}^{\infty} x[n] r_0^{-n}$$

Side by Side

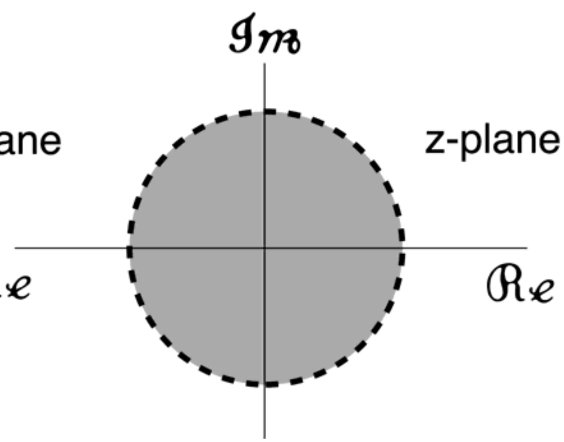
(5) If $x[n]$ is a left-sided sequence, and if $|z| = r_o$ is in the ROC, then all finite values of z for which $0 < |z| < r_o$ are also in the ROC.

(6) If $x[n]$ is two-sided, and if $|z| = r_o$ is in the ROC, then the ROC consists of a ring in the z-plane including the circle $|z| = r_o$.

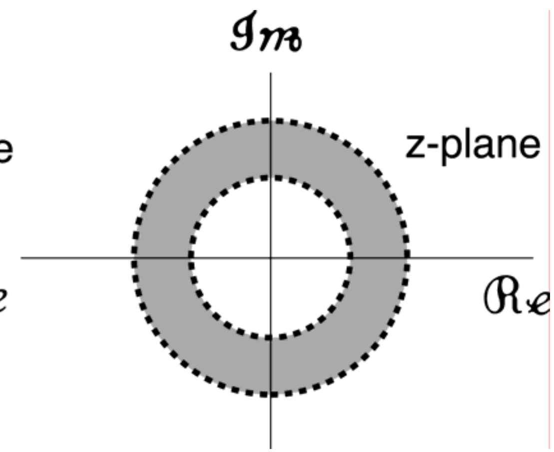
What types of signals do the following ROC correspond to?



right-sided



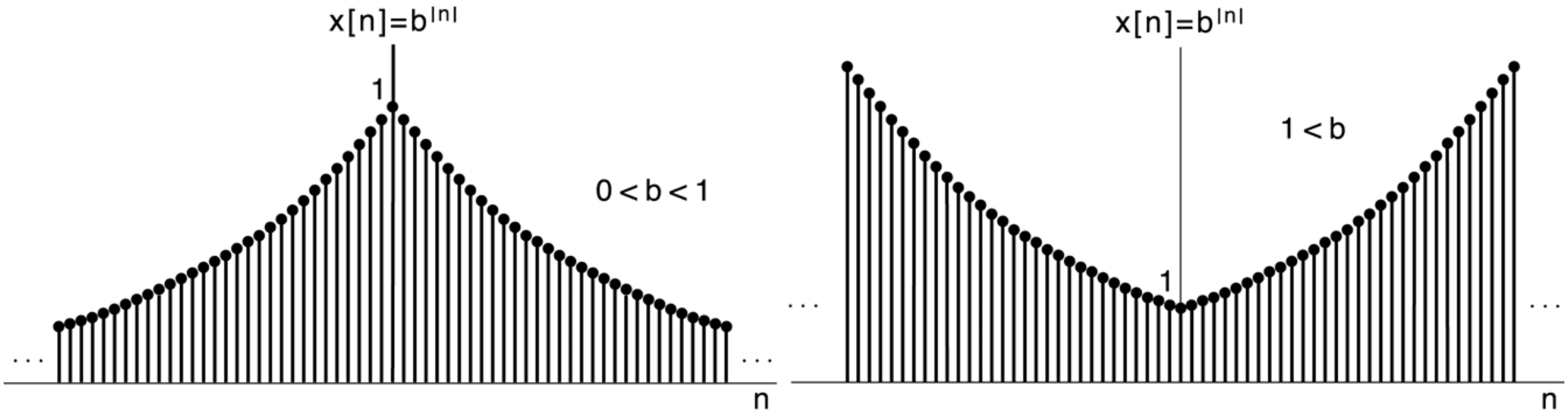
left-sided



two-sided

Example #1

$$x[n] = b^{|n|}, \quad b > 0$$



$$x[n] = b^n u[n] + b^{-n} u[-n-1]$$

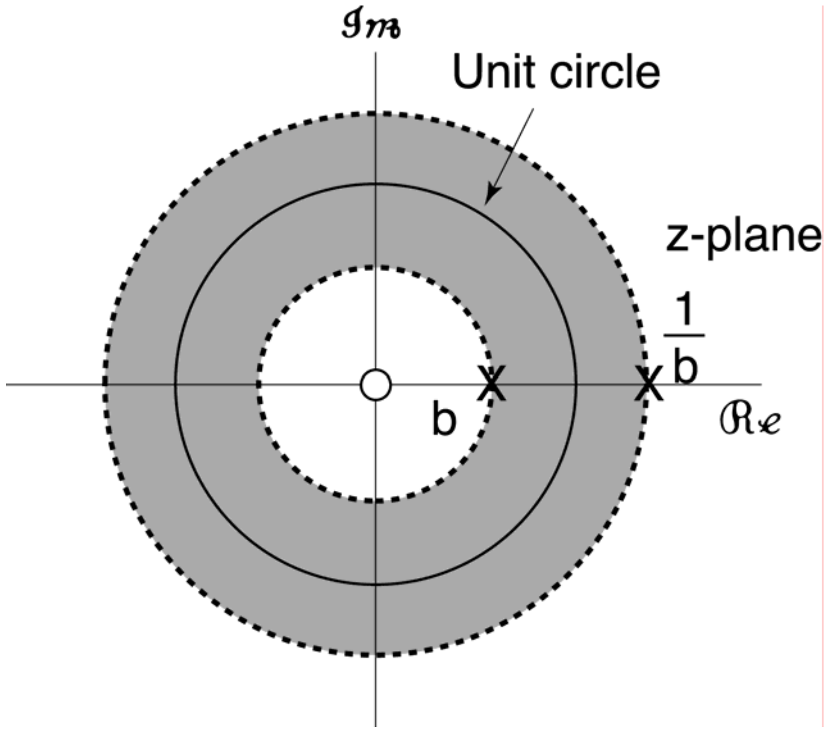
From:

$$b^n u[n] \longleftrightarrow \frac{1}{1 - bz^{-1}}, \quad |z| > b$$

$$b^{-n} u[-n-1] \longleftrightarrow \frac{-1}{1 - b^{-1}z^{-1}}, \quad |z| < \frac{1}{b}$$

Example #1 continued

$$X(z) = \frac{1}{1 - bz^{-1}} + \frac{-1}{1 - b^{-1}z^{-1}} \quad , \quad b < |z| < \frac{1}{b}$$



Clearly, ROC does *not* exist if $b > 1$ $\Rightarrow$ *No* z-transform for $b^{|n|}$.

Inverse z-Transforms

$$X(z) = X(re^{j\omega}) = \mathcal{F}\{x[n]r^{-n}\}, z = re^{j\omega} \in \text{ ROC}$$

$$\Downarrow$$

$$x[n]r^{-n} = \mathcal{F}^{-1}\left\{X(re^{j\omega})\right\} = \frac{1}{2\pi}\int_{2\pi} X(re^{j\omega})e^{j\omega n}d\omega$$

$$\Downarrow$$

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(re^{j\omega})\underbrace{r^n e^{j\omega n}}_{z^n}d\omega$$

for fixed r:

$$z = re^{j\omega} \Rightarrow dz = jre^{j\omega}d\omega \Rightarrow d\omega = \frac{1}{j}z^{-1}dz$$

$$\Downarrow$$

$$x[n] = \frac{1}{2\pi j}\oint X(z)z^{n-1}dz$$

Example #2

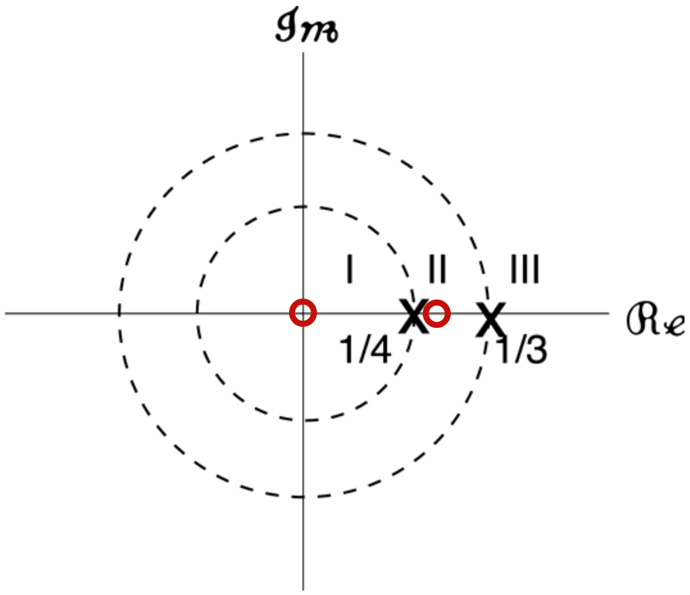
$$X(z) = \frac{3z^2 - \frac{5}{6}z}{(z-\frac{1}{4})(z-\frac{1}{3})} = \frac{3 - \frac{5}{6}z^{-1}}{(1-\frac{1}{4}z^{-1})(1-\frac{1}{3}z^{-1})} = \frac{A}{1-\frac{1}{4}z^{-1}} + \frac{B}{1-\frac{1}{3}z^{-1}}$$

Partial Fraction Expansion Algebra: $A = 1, B = 2$

$$X(z) = \frac{1}{1-\frac{1}{4}z^{-1}} + \frac{2}{1-\frac{1}{3}z^{-1}}$$

$$\updownarrow \qquad \updownarrow \qquad \updownarrow$$

$$x[n] = x_1[n] + x_2[n]$$



zeros at $z = 0$ and
$3z - \frac{5}{6} = 0$ or $z = \frac{5}{18}$

Note, particular to z-transforms:

1) When finding poles and zeros, express $X(z)$ as a function of z.

2) When doing inverse z-transform using PFE, express $X(z)$ as a function of z^{-1}.

ROC III: $|z| > \frac{1}{3}$ - right-sided signal

$$x_1[n] = \left(\frac{1}{4}\right)^n u[n]$$

$$x_2[n] = 2 \cdot \left(\frac{1}{3}\right)^n u[n]$$

ROC II: $\frac{1}{4} < |z| < \frac{1}{3}$ - two-sided signal

$$x_1[n] = \left(\frac{1}{4}\right)^n u[n]$$

$$x_2[n] = -2 \cdot \left(\frac{1}{3}\right)^n u[-n-1]$$

ROC I: $|z| < \frac{1}{4}$ - left-sided signal

$$x_1[n] = -\left(\frac{1}{4}\right)^n u[-n-1]$$

$$x_2[n] = -2 \cdot \left(\frac{1}{3}\right)^n u[-n-1]$$

Inversion by Identifying Coefficients in the Power Series

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$x[n]$ - coefficient of z^{-n}

Example #3: $X(z) = 3z^3 - z + 2z^{-4}$

$$x[-3] = 3$$
$$x[-1] = -1$$
$$x[4] = 2$$
$x[n] = 0$ for all other n's

— A finite-duration DT sequence

Example #4:

(a) $$X(z) = \frac{1}{1 - az^{-1}} = 1 + az^{-1} + (az^{-1})^2 + \cdots$$

$$\Downarrow - \text{ convergent for } |az^{-1}| < 1, \text{i.e.}, |z| > |a|$$

$$x[n] = a^n u[n]$$

(b) $$\begin{aligned} X(z) &= \frac{1}{1 - az^{-1}} = -a^{-1}z\left\{\frac{1}{1 - a^{-1}z}\right\} \\ &= -a^{-1}z(1 + a^{-1}z + (a^{-1}z)^2 + \cdots) \\ &= -a^{-1}z - a^{-2}z^2 - a^{-3}z^3 - \cdots \end{aligned}$$

$$\Downarrow - \text{ convergent for } |a^{-1}z| < 1, \text{i.e.}, |z| < |a|$$

$$x[n] = -a^n u[-n-1]$$

Properties of z-Transforms

(1) Time Shifting $x[n-n_0] \longleftrightarrow z^{-n_0}X(z)$,

The rationality of $X(z)$ unchanged, *different* from LT. ROC unchanged except for the possible addition or deletion of the origin or infinity

$n_o > 0 \Rightarrow$ ROC $z \neq 0$ (maybe)

$n_o < 0 \Rightarrow$ ROC $z \neq \infty$ (maybe)

(2) z-Domain Differentiation $nx[n] \longleftrightarrow -z\dfrac{dX(z)}{dz}$, same ROC

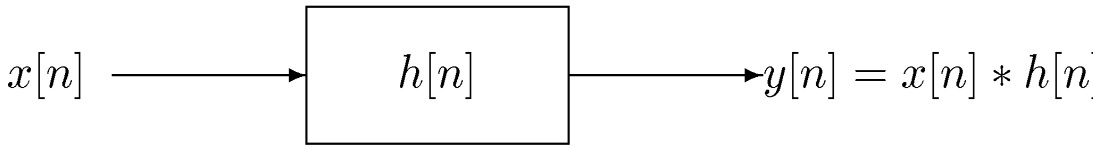
Derivation:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

$$\frac{dX(z)}{dz} = -\sum_{n=-\infty}^{\infty} nx[n]z^{-n-1}$$

$$-z\frac{dX(z)}{dx} = \sum_{n=-\infty}^{\infty} nx[n]z^{-n}$$

Convolution Property and System Functions



$Y(z) = H(z)X(z)$, ROC at least the intersection of the ROCs of $H(z)$ and $X(z)$, can be bigger if there is pole/zero cancellation. *e.g.*

$$
\begin{aligned}
H(z) &= \frac{1}{z-a}, \quad |z| > a \\
X(z) &= z - a, \quad z \neq \infty \\
Y(z) &= 1 \quad \text{ROC all } z
\end{aligned}
$$

$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$$ — The System Function

H(z) + ROC tells us everything about system

CAUSALITY

(1) $h[n]$ right-sided $\Rightarrow$ ROC is the exterior of a circle *possibly* including $z = \infty$:

$$H(z) = \sum_{n=N_1}^{\infty} h[n] z^{-n}$$

If $N_1 < 0$, then the term $h[N_1]z^{-N_1} \rightarrow \infty$ at $z = \infty$
$\Rightarrow$ ROC outside a circle, but does *not* include ∞.

Causal $\Leftrightarrow N_1 \geq 0$

No z^m terms with $m > 0$
$\Rightarrow z = \infty \in$ ROC

$\Downarrow$

A DT LTI system with system function $H(z)$ is causal $\Leftrightarrow$ the ROC of $H(z)$ is the exterior of a circle *including* $z = \infty$

Causality for Systems with Rational System Functions

$$H(z) = \frac{b_M z^M + b_{M-1} z^{M-1} + \cdots + b_1 z + b_0}{a_N z^N + a_{N-1} z^{N-1} + \cdots + a_1 z + a_0}$$

$\Downarrow$ No poles at ∞, if $M \leq N$

A DT LTI system with rational system function $H(z)$ is causal

$\Leftrightarrow$ (a) the ROC is the exterior of a circle outside the outermost pole;

and (b) if we write $H(z)$ as a ratio of polynomials

$$H(z) = \frac{N(z)}{D(z)}$$

then

$$\text{degree } N(z) \leq \text{ degree } D(z)$$

Stability

- LTI System Stable $\Leftrightarrow \sum_{n=-\infty}^{\infty} |h[n]| < \infty \Leftrightarrow$ ROC of $H(z)$ includes the unit circle $|z| = 1$

  $\Rightarrow$ Frequency Response $H(e^{j\omega})$ (DTFT of $h[n]$) exists.

- A causal LTI system with rational system function is stable $\Leftrightarrow$ all poles are inside the unit circle, i.e. have magnitudes < 1