Signals and Systems

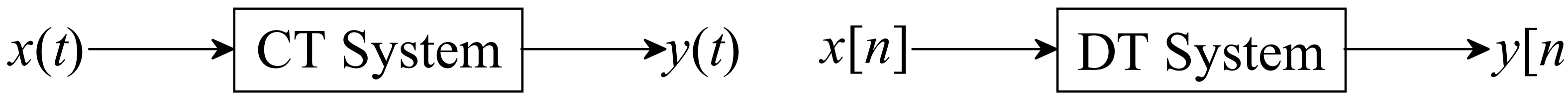
Fall 2003

Lecture #2

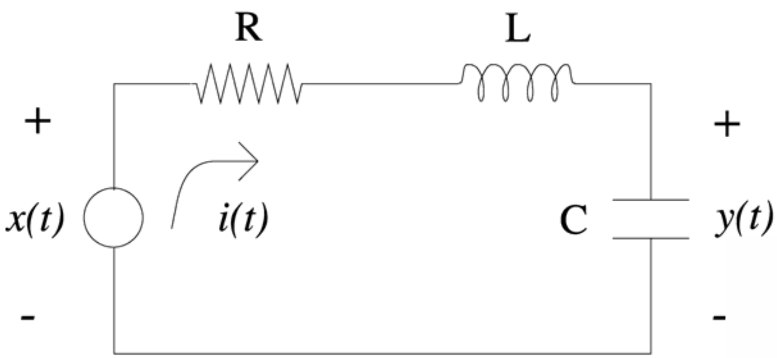
9 September 2003

1) Some examples of systems
2) System properties and examples
   a) Causality
   b) Linearity
   c) Time invariance

SYSTEM EXAMPLES



Ex. #1 RLC circuit



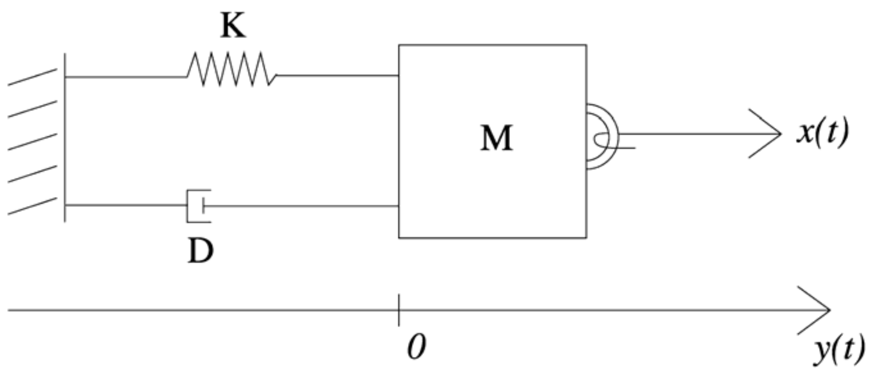
$$\mathrm{R}\ i(t) + \mathrm{L}\frac{di(t)}{dt} + y(t) = x(t)$$

$$i(t) = \mathrm{C}\frac{dy(t)}{dt}$$

$$\Downarrow$$

$$\mathrm{LC}\frac{d^2y(t)}{dt^2} + \mathrm{RC}\frac{dy(t)}{dt} + y(t) = x(t)$$

Ex. #2 Mechanical system



x(t) - applied force
K - spring constant
D - damping constant
y(t) - displacement from rest

Force Balance:

$$\mathrm{M}\frac{d^2y(t)}{dt^2} = x(t) - \mathrm{K}y(t) - \mathrm{D}\frac{dy(t)}{dt}$$

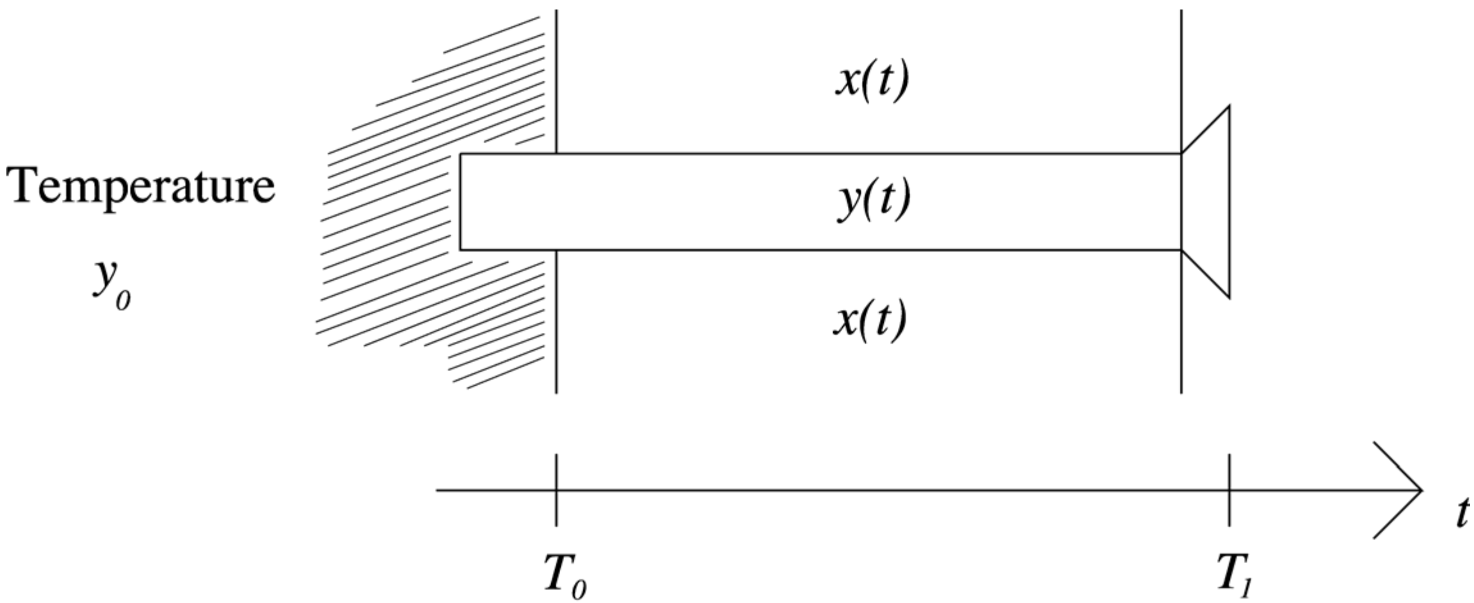
$$\Downarrow$$

$$\mathrm{M}\frac{d^2y(t)}{dt^2} + \mathrm{D}\frac{dy(t)}{dt} + \mathrm{K}y(t) = x(t)$$

Observation: Very different physical systems may be modeled mathematically in very similar ways.

Ex. #3 Thermal system

Cooling Fin in Steady State



t = distance along rod

$y(t)$ = Fin temperature as function of position

$x(t)$ = Surrounding temperature along the fin

Ex. #3 (Continued)

$$\begin{aligned} \frac{d^2y(t)}{dt^2} &= k[y(t) - x(t)] \\ y(T_0) &= y_0 \\ \frac{dy}{dt}(T_1) &= 0 \end{aligned}$$

Observations

- Independent variable can be something other than time, such as space.
- Such systems may, more naturally, have <u>boundary</u> conditions, rather than "initial" conditions.

Ex. #4 Financial system

Fluctuations in the price of zero-coupon bonds

$t = 0$ Time of purchase at price y_0

$t = T$ Time of maturity at value y_T

$y(t)$ = Values of bond at time t

$x(t)$ = Influence of external factors on fluctuations in bond price

$$\frac{d^2y(t)}{dt^2} = f\left(y(t), \frac{dy(t)}{dt}, x_1(t), x_2(t), \ldots, x_N(t), t\right)$$

$$y(0) = y_0, \quad y(T) = y_T.$$

Observation: Even if the independent variable is time, there are interesting and important systems which have boundary conditions.

Ex. #5

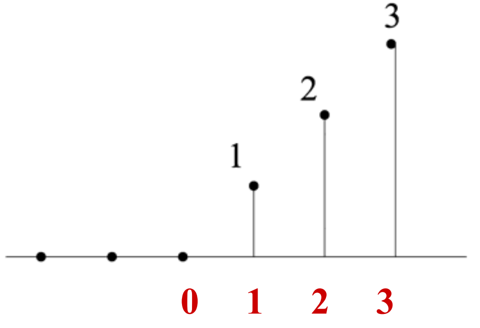
- A rudimentary "edge" detector

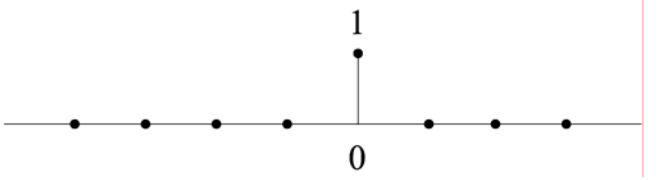
$$
\begin{aligned}
y[n] &= x[n+1] - 2x[n] + x[n-1] \\
&= \{x[n+1] - x[n]\} - \{x[n] - x[n-1]\} \\
&= \text{"Second difference"}
\end{aligned}
$$

- This system detects changes in signal slope

$$(a) \qquad x[n] = n \qquad \Rightarrow \qquad y[n] = 0$$

$$(b) \qquad x[n] = nu[n] \qquad \Rightarrow \qquad y[n]$$





Observations

1) A very rich class of systems (but by no means all systems of interest to us) are described by differential and difference equations.
2) Such an equation, by itself, does not completely describe the input-output behavior of a system: we need auxiliary conditions (initial conditions, boundary conditions).
3) In some cases the system of interest has time as the natural independent variable and is causal. However, that is not always the case.
4) Very different physical systems may have very similar mathematical descriptions.

SYSTEM PROPERTIES
(Causality, Linearity, Time-invariance, etc.)

WHY ?

- Important practical/physical implications

- They provide us with insight and structure that we can exploit both to analyze and understand systems more deeply.

CAUSALITY

- A system is causal if the output does not anticipate future values of the input, i.e., if the output at any time depends only on values of the input up to that time.

- All real-time physical systems are causal, because time only moves forward. Effect occurs after cause. (Imagine if you own a noncausal system whose output depends on tomorrow's stock price.)

- Causality does not apply to spatially varying signals. (We can move both left and right, up and down.)

- Causality does not apply to systems processing recorded signals, e.g. taped sports games vs. live broadcast.

CAUSALITY (continued)

- Mathematically (in CT): A system $x(t) \rightarrow y(t)$ is causal if

when $x_1(t) \rightarrow y_1(t) \quad x_2(t) \rightarrow y_2(t)$

and $x_1(t) = x_2(t)$ for all $t \leq t_o$

Then $y_1(t) = y_2(t)$ for all $t \leq t_o$

CAUSAL OR NONCAUSAL

$y(t) = x^2(t-1)$

E.g. $y(5)$ depends on $x(4)$ ... causal

$y(t) = x(t+1)$

E.g. $y(5) = x(6), y$ depends on future $\Rightarrow$ noncausal

$y[n] = x[-n]$

E.g. $y[5] = x[-5]$ ok, but

$y[-5] = x[5], y$ depends on future $\Rightarrow$ noncausal

$y[n] = \left(\frac{1}{2}\right)^{n+1} x^3[n-1]$

E.g. $y[5]$ depends on $x[4]$ ... causal

TIME-INVARIANCE (TI)

Informally, a system is time-invariant (TI) if its behavior does not depend on what time it is.

- Mathematically (in DT): A system $x[n] \rightarrow y[n]$ is TI if for any input $x[n]$ and any time shift n_0,

If $$x[n] \rightarrow y[n]$$

then $$x[n - n_0] \rightarrow y[n - n_0] \,.$$

- Similarly for a CT time-invariant system,

If $$x(t) \rightarrow y(t)$$

then $$x(t - t_o) \rightarrow y(t - t_o) \,.$$

TIME-INVARIANT OR TIME-VARYING ?

$$y(t) = x^2(t+1)$$

TI

$$y[n] = \left(\frac{1}{2}\right)^{n+1} x^3[n-1]$$

Time-varying (NOT time-invariant)

NOW WE CAN DEDUCE SOMETHING!

Fact: If the input to a TI System is periodic, then the output is periodic with the same period.

"Proof": Suppose $x(t+T) = x(t)$

and $x(t) \rightarrow y(t)$

Then by TI

$$x(t+T) \rightarrow y(t+T).$$

$\uparrow$ These are the same input!

$\uparrow$ So these must be the same output, i.e., $y(t) = y(t+T)$.

LINEAR AND NONLINEAR SYSTEMS

- Many systems are nonlinear. For example: many circuit elements (e.g., diodes), dynamics of aircraft, econometric models,…
- However, in 6.003 we focus exclusively on **linear** systems.
- Why?
  - Linear models represent accurate representations of behavior of many systems (e.g., linear resistors, capacitors, other examples given previously,…)
  - Can often linearize models to examine "small signal" perturbations around "operating points"
  - Linear systems are analytically tractable, providing basis for important tools and considerable insight

LINEARITY

A (CT) system is linear if it has the superposition property:

If $x_1(t) \rightarrow y_1(t)$ and $x_2(t) \rightarrow y_2(t)$

then $ax_1(t) + bx_2(t) \rightarrow ay_1(t) + by_2(t)$

$\mathrm{y}[n] = x^2[n]$ Nonlinear, TI, Causal

$y(t) = x(2t)$ Linear, <u>not</u> TI, Noncausal

Can you find systems with other combinations ?

- e.g. Linear, TI, Noncausal
  Linear, not TI, Causal

PROPERTIES OF LINEAR SYSTEMS

- Superposition

If $$x_k[n] \longrightarrow y_k[n]$$

Then $$\sum_k a_k x_k[n] \longrightarrow \sum_k a_k y_k[n]$$

- For linear systems, zero input $\rightarrow$ zero output

"Proof" $$0 = 0 \cdot x[n] \rightarrow 0 \cdot y[n] = 0$$

Properties of Linear Systems (Continued)

- A linear system is causal if and only if it satisfies the condition of initial rest:

$$x(t) = 0 \text{ for } t \leq t_0 \rightarrow y(t) = 0 \text{ for } t \leq t_0 \ (*).$$

"Proof"

a) Suppose system is causal. Show that (*) holds.

b) Suppose (*) holds. Show that the system is causal.

LINEAR TIME-INVARIANT (LTI) SYSTEMS

- Focus of <u>most</u> of this course

  - Practical importance (Eg. #1-3 earlier this lecture are all LTI systems.)

  - The powerful analysis tools associated with LTI systems

- A basic fact: If we know the response of an LTI system to some inputs, we actually know the response to ***many*** inputs

Example: DT LTI System