

6.003: Signals and Systems—Fall 2003

PROBLEM SET 10 SOLUTIONS

Home study exercise (E1) O&W 11.32

(a)

$$\begin{aligned} \frac{Y(s)}{X(s)} &= \frac{H(s)}{1 + KG(s)H(s)} \quad (\text{via Black's equation}) \\ &= \frac{N_1(s)D_2(s)}{D_1(s)D_2(s) + KN_1(s)N_2(s)} \quad (\text{multiply through by } D_1(s)D_2(s)) \end{aligned}$$

The zeros of this closed-loop system are the roots of N_1 (the zeros of H) and the roots of D_2 (the poles of G).

(b) If $K = 0$, then the system is operating without feedback of any sort. Naturally the poles and zeros of the system without feedback are the poles and zeros of $H(s)$.

More formally, we can take limits of the above equation:

$$\lim_{K \rightarrow 0} \frac{N_1(s)D_2(s)}{D_1(s)D_2(s) + KN_1(s)N_2(s)} = \frac{N_1(s)D_2(s)}{D_1(s)D_2(s)} = \frac{N_1(s)}{D_1(s)} = H(s)$$

(c) Check by simple substitution:

$$\begin{aligned} Q(s) &= \frac{p(s)}{q(s)} \cdot \left[\frac{\hat{H}(s)}{1 + K\hat{G}(s)\hat{H}(s)} \right] \\ &= \frac{p(s)}{q(s)} \cdot \left[\frac{\frac{N_1(s)/p(s)}{D_1(s)/q(s)}}{1 + K \cdot \frac{N_2(s)/q(s)}{D_2(s)/p(s)} \cdot \frac{N_1(s)/p(s)}{D_1(s)/q(s)}} \right] \\ &= \frac{H(s)}{1 + KG(s)H(s)} \end{aligned}$$

(d) We are given the following:

$$H(s) = \frac{s+1}{(s+4)(s+2)}, \quad G(s) = \frac{s+2}{s+1}$$

In this simple example, we can find p and q by inspection:

$$p(s) = s+1, \quad q(s) = s+2, \quad \hat{H}(s) = \frac{1}{s+4}, \quad \hat{G}(s) = 1$$

$$\text{Root locus equation: } \hat{H}(s)\hat{G}(s) = \frac{1}{s+4} = -\frac{1}{K} \implies s = -(K+4)$$

The closed-loop system has one finite zero ($s = -1$), one “zero at infinity”, one pole which is *not* affected by K ($s = -2$), and one pole which is affected by K ($s = -(K + 4)$).

Note that the zeros in the feedback system can *obscure* certain complex frequencies in the output; these frequencies will never show up in the error signal. This is another reason to avoid pole-zero cancellation as a design technique, since it can cause *unstable* complex frequencies to become unobservable in the error signal.

Problem 1

Let the output of $G(s)$ be $y(t)$. Then, we can obtain the transfer function from $x(t)$ to $y(t)$ using Black’s formula:

$$\frac{Y(s)}{X(s)} = \frac{KG(s)}{1 + KG(s)}.$$

Thus, the closed loop poles are s satisfying $1 + KG(s) = 0$. To plot a root locus, you are sketching the location of s as a function of K such that $1 + KG(s) = 0$.

Having said that, all the points on the locus satisfy the following criteria usually referred to as the angle criteria:

$$\begin{aligned} 1 + KG(s_l) &= 0 \\ G(s_l) &= -\frac{1}{K}, \end{aligned} \tag{1}$$

where s_l is on the locus. If $K > 0$, then $\angle G(s_l)$ be an odd integer multiple of π , while for $K < 0$, then $\angle G(s_l)$ be an even integer multiple of π .

Also, from Eqn(1), it is easy to see that s_l approaches to the poles of $G(s)$ if $|K| \rightarrow 0$ and that s_l approaches to the zeros of $G(s)$ if $|K| \rightarrow \infty$. One thing to note here, however, is that by closing the loop in the feedback configuration shown in the problem, the number of the closed loop poles is either the same as that of $G(s)$ or less if pole-zero cancellation takes place. Thus, to make the argument above applicable to any rational transfer functions, we need to have the same number of poles and zeros for $G(s)$. This leads us to define poles or zeros at ”infinity” as mentioned in the Home study exercise above. With all these concepts in your mind, let’s go through the questions.

(a) Given:

$$G(s) = \frac{1}{s + 1}.$$

$G(s)$ has a pole at -1 and no finite zero.

- For $K > 0$: In this case, with the angle criteria, all the points, s_l on the locus should satisfy

$$\angle G(s_l) = \text{odd integer multiple of } \pi.$$

So, if you take any point on the real line segment $(-\infty, -1)$,

$$\angle G(s_l) = \underbrace{0}_{\text{phase due to zeros}} - \underbrace{\pm\pi}_{\text{phase due to the pole}} = \pm\pi.$$

Also, this is the only segment that belongs to the locus.

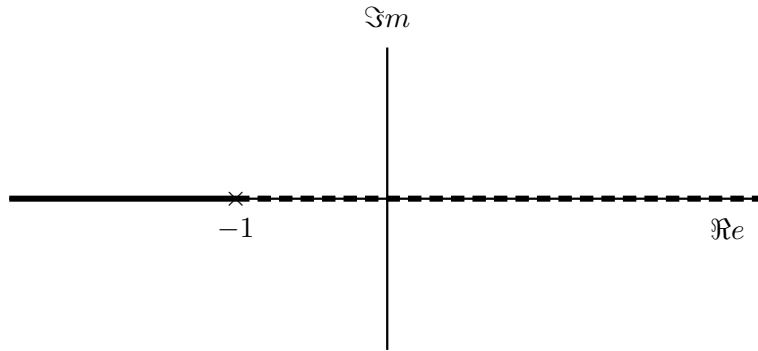
- For $K < 0$: The angle criteria states that:

$$\angle G(s_l) = \text{even integer multiple of } \pi.$$

So, if you take any point on the real line segment $(-1, \infty)$,

$$\angle G(s_l) = \underbrace{0}_{\text{phase due to zeros}} - \underbrace{0}_{\text{phase due to the pole}} = 0.$$

Combining the two cases, the root loci for both cases are plotted as shown below. The solid line corresponds to the locus for $K > 0$ case and the dashed line for $K < 0$ case.



(b) Given:

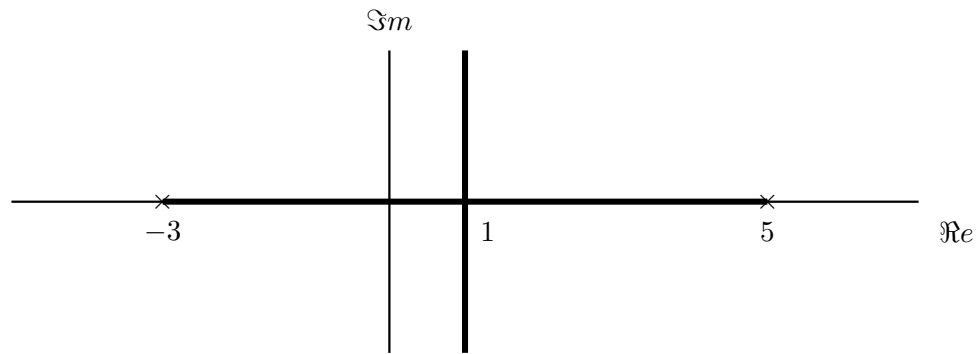
$$G(s) = \frac{1}{(s-5)(s+3)}.$$

$G(s)$ has two poles at 5 and -3 , and no finite zero.

- For $K > 0$: Evoking the angle criteria, we can first see that the real line segment $(-3, 5)$ belongs to the locus. Now we would like to know at which point the locus branches out or bifurcates from the real line. At the branching point, the closed loop poles are of double pole. To find such a point, we can first find a value of K corresponding to the double pole.

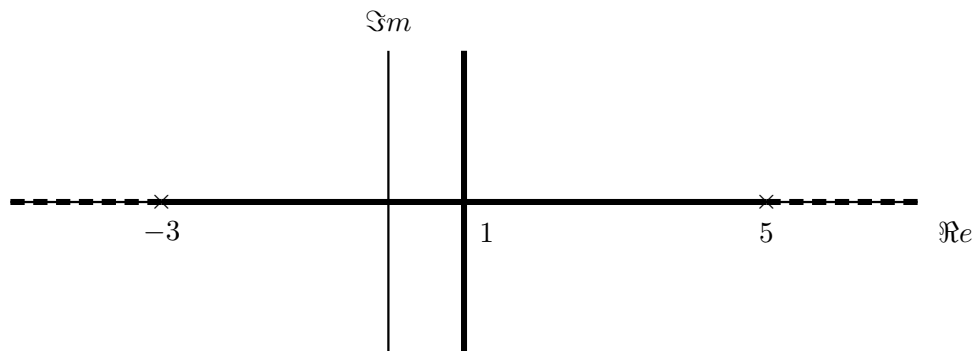
$$\begin{aligned} 1 + KG(s) &= 0 \\ 1 + \frac{K}{(s-5)(s+3)} &= 0 \\ (s-5)(s+3) + K &= 0 \\ s^2 - 2s - 15 + K &= 0 \\ (s-1)^2 - 1 - 15 + K &= 0 \quad \text{completing the square.} \end{aligned}$$

So, when $-1 - 15 + K = 0$ or $K = 16$, the closed loop system has a double pole at 1. Because of the symmetry of the poles about $s = 1$, the locus branches out from $s = 1$ along $\Re\{s\} = 1$ parallel to the imaginary axis. This can be seen in the figure below:



- For $K < 0$: In this case, using the angle criteria, we can see that the real line segment $(5, \infty)$ belongs to the locus since both poles contribute 0 or even integer multiple of π . Also, the real line segment $(-\infty, -3)$ belongs to the locus since both poles contribute π or odd integer multiple of π ; the net phase of $G(s)$ is even integer multiple of π .

Thus, the root loci containing both $K > 0$ (solid line) and $K < 0$ (dashed line) cases are shown below:



(c) Given:

$$G(s) = \frac{s+1}{s^2}.$$

$G(s)$ has a double-pole at 0 and one finite zero at -1 .

- For $K > 0$: With the angle criteria, we can see that the real line segment $(-\infty, -1)$ belongs to the locus. Since both poles of $G(s)$ do not belong to the segment, there will be a double-pole point in the segment. To find where it is, use the same technique as in (b),

i.e., complete the square and find the corresponding gain K and the bifurcation point.

$$\begin{aligned}
 1 + KG(s) &= 0 \\
 1 + K \frac{s+1}{s^2} &= 0 \\
 s^2 + Ks + K &= 0 \\
 \left(s + \frac{K}{2}\right)^2 + K - \left(\frac{K}{2}\right)^2 &= 0 \\
 \rightarrow K \left(1 - \frac{K}{4}\right) &= 0 \\
 \therefore K &= 0, 4.
 \end{aligned}$$

This result tells us that at $K = 0$ and $K = 4$ we have double-pole and the corresponding locations are at $s = 0$ and $s = -\frac{4}{2} = -2$.

How can we know the shape of the path that locus takes after branching out of $s = 0$ and merging at $s = -2$? In the case in our hand, it is not hard to determine. Since the closed loop poles for the K between 0 and 4 are a complex conjugate pair, we can assume that $s = \sigma + j\omega$ for a given K . Then, using one of the equations above;

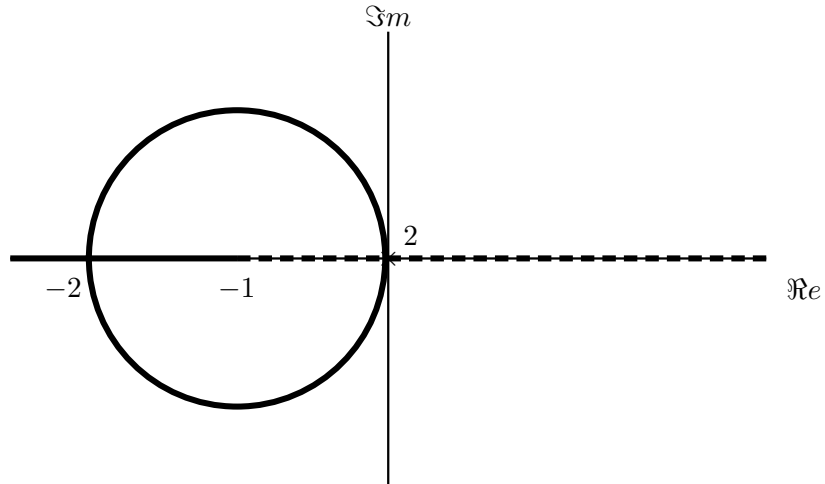
$$\begin{aligned}
 (\sigma + j\omega)^2 + K(\sigma + j\omega) + K &= 0 \\
 (\sigma^2 + 2j\sigma\omega - \omega^2) + K\sigma + jK\omega + K &= 0 \\
 \text{For real part: } \sigma^2 - \omega^2 + K\sigma + K &= 0 \\
 \text{For imaginary part: } 2\sigma\omega + K\omega &= 0 \rightarrow K = -2\sigma \\
 \sigma^2 - \omega^2 - 2\sigma^2 - 2\sigma &= 0 \quad (\text{combining the two equations above}) \\
 \sigma^2 + \omega^2 + 2\sigma &= 0 \\
 (\sigma + 1)^2 + \omega^2 &= 1,
 \end{aligned}$$

As you can see, the last equation is nothing but an equation of a circle whose radius is 1 and whose center is located at $(\sigma, \omega) = (-1, 0)$. Thus, the locus is of the circle for $0 < K < 4$. Then, at $s = -2$ the locus bifurcates to the zero at $s = -1$ and $s = -\infty$ on the real axis.

One thing to note here is that, in general, it is extremely difficult to obtain expressions of the locus for higher order systems.

- For $K < 0$: In this case, we can see that the real line segments $(-1, 0)$ and $(0, \infty)$ belong to the locus. On the locus, one of the poles will move to the zero at $s = -1$ as K decreases or $-K$ increases, while the other pole moves out to ∞ along the positive real axis.

Thus, the loci containing both $K > 0$ (solid) and $K < 0$ (dashed) cases are shown below:



Problem 2

- (a) First let output of the plant $G(s) = \frac{1}{s(s+10)}$ be $y(t)$. Then, by Black's formula, the closed transfer function from the input $x(t)$ to $y(t)$ can be found as:

$$\begin{aligned} \frac{Y(s)}{X(s)} &= \frac{G(s)K(s)}{1 + G(s)K(s)} \\ &= \frac{\frac{K}{s(s+10)}}{1 + \frac{K}{s(s+10)}} \\ \therefore \frac{Y(s)}{X(s)} &= \frac{K}{s(s+10) + K}. \end{aligned} \quad (2)$$

The error signal $e(t)$ is the difference between $x(t)$ and $y(t)$. Thus from the linearity of Laplace transforms:

$$\begin{aligned} e(t) &= x(t) - y(t) \\ E(s) &= X(s) - Y(s). \end{aligned} \quad (3)$$

So, with Eqns (2) and (3), we can obtain the transfer function from the input $x(t)$ to the error $e(t)$:

$$\begin{aligned}
\frac{E(s)}{X(s)} &= 1 - \frac{Y(s)}{X(s)} \quad \because \text{Eqn (3)} \\
&= 1 - \frac{K}{s(s+1)+K} \quad \because \text{Eqn (2)} \\
&= \frac{s(s+10)}{s(s+10)+K} \\
\therefore \frac{E(s)}{X(s)} &= \frac{s(s+10)}{s^2+10s+K}. \tag{4}
\end{aligned}$$

Thus, using Eqn (4), $E(s)$ for the unit step input $x(t) = u(t)$ is:

$$\begin{aligned}
E(s) &= \frac{s(s+10)}{s^2+10s+K} X(s) \\
&= \frac{s(s+10)}{s^2+10s+K} \frac{1}{s} \\
&= \frac{s+10}{s^2+10s+K},
\end{aligned}$$

so if $K \leq 0$, we can see from Routh-Hurwitz criteria that $E(s)$ has at least one of the poles in the right half plane, i.e, the real part of the pole is positive. Hence, in this case, the steady state tracking error diverges to ∞ .

Now, consider the case when $K > 0$. Then, both poles of $E(s)$ are in the left half plane and the order of the numerator is strictly less than that of the denominator (the latter condition is usually referred to as being strictly proper) . Thus we can apply the final value theorem to compute the steady state tracking error:

$$\begin{aligned}
e(\infty) &= \lim_{t \rightarrow \infty} e(t) \\
&= \lim_{s \rightarrow 0} sE(s) \\
&= \lim_{s \rightarrow 0} s \frac{s+10}{s^2+10s+K} \\
&= 0.
\end{aligned}$$

Thus, regardless of the value of K , as far as it is positive, $e(\infty) = 0$.

(b) To compute the error signal to the ramp input, we can use Eqn(4) with now $X(s) = \frac{1}{s^2}$:

$$\begin{aligned}
E(s) &= \frac{s(s+10)}{s^2+10s+K} \frac{1}{s^2} \\
&= \frac{s+10}{s(s^2+10s+K)} \\
&= \underbrace{\frac{10}{s}}_{E_1(s)} + \underbrace{\frac{-\frac{10}{K}s+1-\frac{100}{K}}{s^2+10s+K}}_{E_2(s)}.
\end{aligned}$$

$e_1(t)$, the corresponding time signal to $E_1(s)$ is a scaled step. $E_2(s)$ has two poles and they remain in the left-half plane as long as $K > 0$. In this case $e_2(t)$, the corresponding time signal to $E_2(s)$ will decay to 0 as $t \rightarrow \infty$. Thus, $e(\infty) = \frac{10}{K}$, i.e., the steady state error will not be 0. On the other hand, if $K \leq 0$. Then, $E_2(s)$ now has at least one of the poles in the right-half plane. Thus, $e_2(\infty) \rightarrow \infty$ as $t \rightarrow \infty$; therefore $e(\infty) \rightarrow \infty$.

- (c) For this part, first we would like to get the transfer function from $x(t)$ to $y(t)$. Let's denote the output of K_f , $x_f(t)$, then $X_f(s) = K_f X(s)$.

$$\begin{aligned} Y(s) &= \frac{1}{s(s+10)}(X_f(s) + K_s X(s) - Y(s)) \\ \left(1 + \frac{1}{s(s+10)}\right) Y(s) &= \frac{1}{s(s+10)}(K_f + K_s)X(s) \\ \frac{Y(s)}{X(s)} &= \frac{\frac{1}{s(s+10)}(K_f + K_s)}{1 + \frac{1}{s(s+10)}} \\ \therefore \frac{Y(s)}{X(s)} &= \frac{K_f + K_s}{s(s+10) + 1}. \end{aligned}$$

Since $e(t) = x(t) - y(t)$, we can compute the Laplace transform of $e(t)$, $E(s)$ as follows:

$$\begin{aligned} E(s) &= X(s) - Y(s) \\ E(s) &= \left(1 - \frac{Y(s)}{X(s)}\right) X(s). \end{aligned}$$

The input $x(t)$ to the system is a ramp, so $X(s) = \frac{1}{s^2}$. Thus, $E(s)$ is:

$$\begin{aligned} E(s) &= \left(1 - \frac{K_f + K_s}{s(s+10) + 1}\right) \frac{1}{s^2} \\ &= \frac{s^2 + 10s + 1 - (K_f + K_s)}{s^2(s^2 + 10s + 1)} \\ \rightarrow sE(s) &= \frac{s^2 + 10s + 1 - (K_f + K_s)}{s(s^2 + 10s + 1)}. \end{aligned}$$

To be able to apply the final value theorem, we want to choose $K_f(s)$ and $K_s(s)$ such that the poles of $sE(s)$ are in the left-half plane. In addition, we need to choose $K_f(s)$ and $K_s(s)$ such that

$$\lim_{s \rightarrow 0} sE(s) = 0$$

to ensure that the steady state error to the ramp input is zero. Combining these two conditions, we see that we need to make the numerator of $sE(s)$ equal to s^2 . From this conclusion alone, we only know that $K_f(s) + K_s(s) = 10s + 1$. Assuming that neither is 0, then we can choose

$K_f(s) = 10s$ and $K_s = 1$. Then,

$$\begin{aligned} sE(s) &= \frac{s^2 + 10s + 1 - (10s + 1)}{s(s^2 + 10s + 1)} \\ &= \frac{s^2}{s(s^2 + 10s + 1)} \\ \therefore sE(s) &= \frac{s}{s^2 + 10s + 1}. \end{aligned}$$

Now, we can safely apply the final value theorem to obtain the steady state error and we indeed obtain 0 as expected:

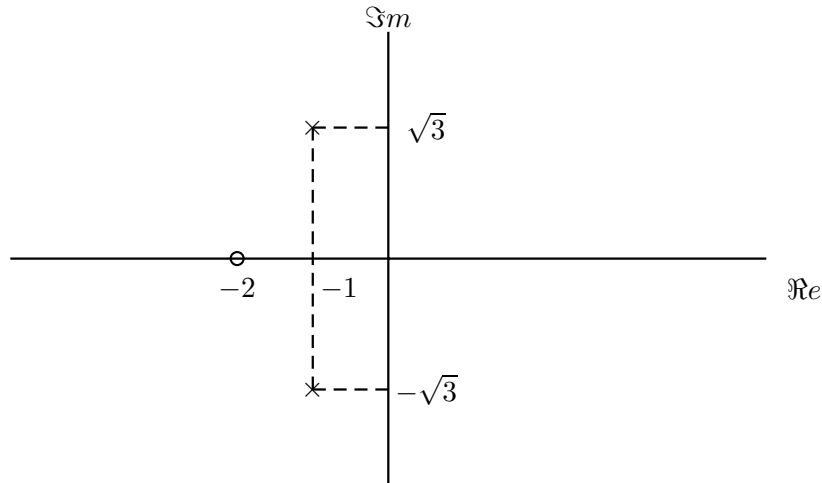
$$\lim_{s \rightarrow 0} sE(s) = \lim_{s \rightarrow 0} \frac{s}{s^2 + 10s + 1} = 0.$$

Problem 3 (O&W 11.27)

Given:

$$H(s) = \frac{s + 2}{s^2 + 2s + 4}, \quad G(s) = K.$$

First, let's identify the poles and zeros of $H(s)$. Since the system $H(s)$ is a second order system, there are two poles and two zeros. It is clear that one of the zeros is of finite at -2 and the other is at ∞ . The poles are the solutions of $s^2 + 2s + 4 = 0$, so $-1 \pm \sqrt{3}$. Thus, the pole-zero diagram is as shown below:



Since $H(s)$ is causal, the ROC for $H(s)$ is $\Re\{s\} > -1$.

The closed loop poles are the solution to the following equation:

$$1 + KH(s) = 1 + K \frac{s + 2}{s^2 + 2s + 4} = 0,$$

or

$$s^2 + 2s + 4 + K(s + 2) = s^2 + (K + 2)s + 2K + 4 = 0. \quad (5)$$

- (a) For $K > 0$, for all s_l on the locus $\angle H(s_l)$ is an odd integer multiple of π . Thus, we can first see that the real line segment $(-\infty, -2)$ belongs to the locus. Then, we would like to know at which point on that segment there is a double-pole. On that point, Eqn (5) has a double root, i.e.,

$$\begin{aligned} s^2 + (K + 2)s + 2K + 4 &= 0 && \text{completing the square} \\ \left(s + \frac{K + 2}{2}\right)^2 - \left(\frac{K + 2}{2}\right)^2 + 2K + 4 &= 0 \\ \rightarrow -\left(\frac{K + 2}{2}\right)^2 + 2K + 4 &= 0 \\ \rightarrow K &= -2, 6. \end{aligned}$$

This tells us that there are two points where double pole occurs; one is when $K = 6 > 0$ and the other is when $K = -2 < 0$. Here, we need to look at only the positive case. The negative one will be used in part (b). For $K = 6$, the double-root is at $-\frac{K+2}{2} = -4$.

- (b) For $K < 0$, using the angle criteria, we can see that the real line segment $(-2, \infty)$ belongs to the locus and the double-pole point is at $-\frac{K+2}{2} = 0$ where $K = -2$ as computed in (a).
- (c) Since the closed-loop system is still just a second order system, the condition that the closed-loop impulse response does not exhibit any oscillatory behavior simply means that the closed-loop system is critically damped or overdamped. Since $H(s)$ is underdamped i.e., its poles are not on the real axis, at the smallest K to be found, the denominator takes the following form:

$$\begin{aligned} s^2 + 2\xi\omega_n s + \omega_n^2 &= 0 \\ s^2 + 2\omega_n s + \omega_n^2 &= 0, && \xi = 1 \quad \text{critically damped} \\ (s + \omega_n)^2 &= 0, \end{aligned}$$

where ω_n is the undamped natural frequency of the closed loop system. Thus, the required condition is nothing but the double-pole condition we found in (a). Since K is constrained to be positive, the smallest K we are looking for is $K = 6$.

Problem 4

Note that z -transform is linear.

(a) Given:

$$x[n] = 2\delta[n + 3] - \delta[n - 2].$$

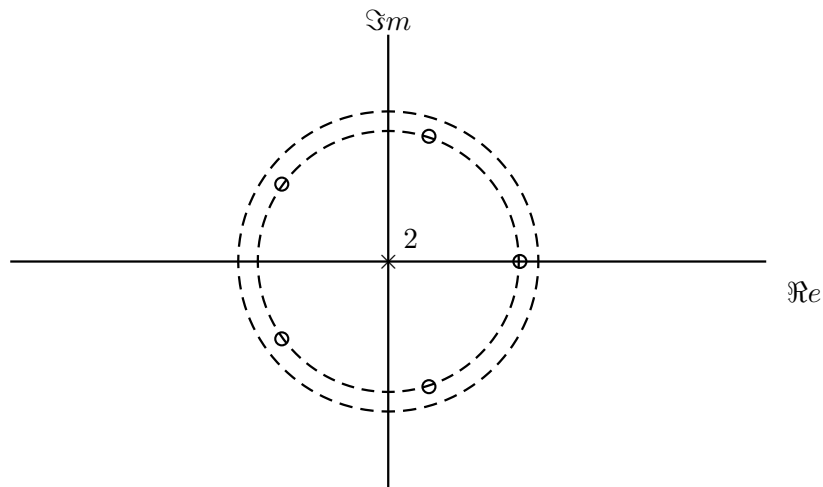
In this part of the problem, we would like to use one of the most fundamental z -transform pairs:

$$\delta[n - m] \xleftrightarrow{z} z^{-m}, \quad \text{ROC: All } z \text{ except for } 0 \text{ if } m > 0 \text{ or } \infty \text{ if } m < 0. \quad (6)$$

By applying this pair for both terms in the given $x[n]$, we get:

$$\begin{aligned} X(z) &= 2z^3 - z^{-2} \\ &= \frac{2z^5 - 1}{z^2}, \end{aligned}$$

with its ROC is all z except for 0 and ∞ . Since the unit circle is included in the ROC, the Fourier transform of the sequence exists. From the last expression, it can be seen that there are two poles both of which are at 0. There are five zeros and they are the solutions of $2z^5 - 1 = 0$, i.e, $2^{-\frac{1}{5}}e^{j\omega_z}$ where ω_z is integer multiples of $\frac{2\pi}{5}$.



To see this, remember that any complex number z can be written in polar form as $z = re^{j\omega}$ where r is the length, or radius of the vector z and ω is its phase. In our case using this idea, we can find the zeros as follows:

$$\begin{aligned}
2z^5 - 1 &= 0 \\
z^5 &= \frac{1}{2} \\
r^5 e^{j5\omega_z} &= \frac{1}{2} \\
&\downarrow \\
r &= \left(\frac{1}{2}\right)^{\frac{1}{5}} = 2^{-\frac{1}{5}} \\
j5\omega_z &= \text{integer multiple of } 2\pi.
\end{aligned}$$

The pole-zero diagram is depicted above. The 5 zeros are on the inner circle whose radius is $2^{-1/5}$ and the outer circle is a unit circle. 2 at the origin denotes that there is a double pole at the origin. The ROC is everywhere except for the origin and ∞ .

(b) Given:

$$x[n] = \underbrace{2^n u[n-1]}_{x_1[n]} + \underbrace{4^n u[-n]}_{x_2[n]}.$$

In this part, we would like to use another crucial z -transform pair:

$$\begin{cases} a^n u[n], & |z| > |a| \\ -a^n u[-n-1], & |z| < |a| \end{cases} \xleftrightarrow{z} \frac{1}{1 - az^{-1}}. \quad (7)$$

Note that

$$x_1[n] = 2^n u[n-1] = 2 \cdot 2^{n-1} u[n-1],$$

and

$$x_2[n] = 4^n u[-n] = 4^n u[-n-1] + \delta[n].$$

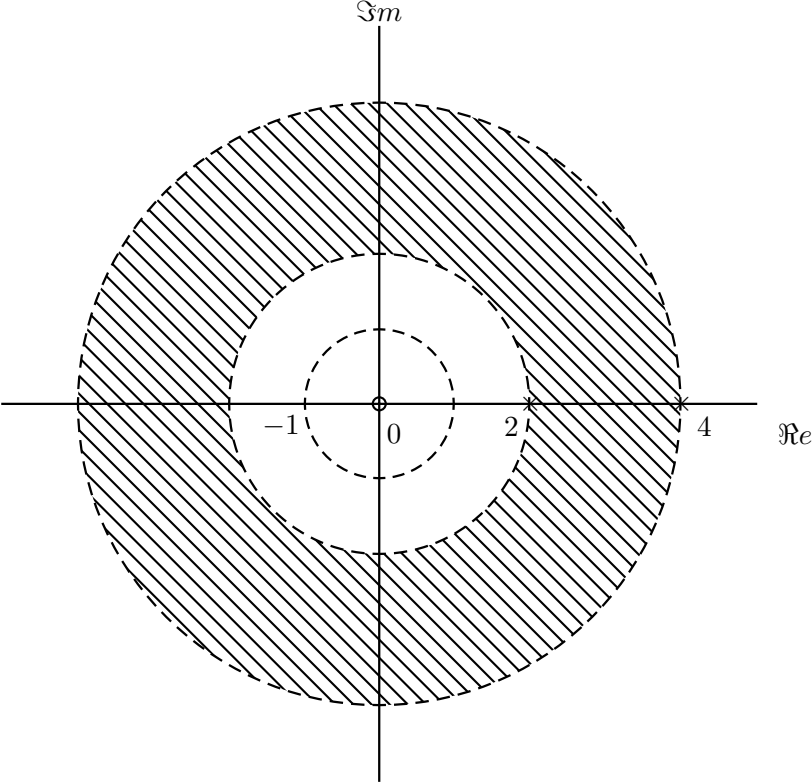
Let $X_1(z)$ and $X_2(z)$ be z -transforms of $x_1[n]$ and $x_2[n]$ respectively. Then, by applying Eqns (6) and (7) to $x_1[n]$ and $x_2[n]$, we have:

$$\begin{aligned}
X_1(z) &= 2 \cdot \frac{1}{1 - 2z^{-1}} \cdot z^{-1} = \frac{2}{z-2}, \quad |z| > 2 \\
X_2(z) &= - \cdot \frac{1}{1 - 4z^{-1}} + 1 = \frac{-4}{z-4}, \quad |z| < 4.
\end{aligned}$$

Thus,

$$\begin{aligned}
X(z) &= X_1(z) + X_2(z) \\
&= \frac{2}{z-2} - \frac{4}{z-4} \\
\therefore X(z) &= - \frac{2z}{(z-2)(z-4)}
\end{aligned}$$

with its ROC being an intersection of $|z| > 2$ and $|z| < 4$, i.e., $2 < |z| < 4$. The ROC does not contain the unit circle; thus the Fourier transform of the sequence does not exist. There are two poles at 2 and 4 and is one zero at 0. Thus, the pole-zero plot with the ROC is depicted below:



One thing to note from Problem 4 is that it would be better to change the expression of z -transforms into rational functions in z to find poles and zeros, while it would be better to change the expressions into rational functions of z^{-1} to find the corresponding time sequences since in many cases z -transform tables show almost all the transforms in terms of z^{-1} such as the one on p.776 in O&W.

Problem 5

(a) Recall the z -transform pair:

$$z^{-m} \xleftrightarrow{z} \delta[n - m].$$

Thus, we need to simply apply this to each term in the given expression. The answer therefore is:

$$x[n] = 12\delta[n - 4] - \delta[n - 1] + 6\delta[n] + 9\delta[n + 2] - 8\delta[n + 5].$$

(b) Given:

$$X(z) = \frac{5}{1 + \frac{1}{6}z^{-1} - \frac{1}{6}z^{-2}}, \quad \frac{1}{3} < |z| < \frac{1}{2}.$$

First, we would like to perform partial fraction to reduce $X(z)$ consisting of easily recognizable terms:

$$\begin{aligned} X(z) &= \frac{5}{1 + \frac{1}{6}z^{-1} - \frac{1}{6}z^{-2}} \\ &= \frac{5}{\left(1 - \frac{1}{3}z^{-1}\right)\left(1 + \frac{1}{2}z^{-1}\right)} \\ &= \frac{2}{\left(1 - \frac{1}{3}z^{-1}\right)} + \frac{3}{\left(1 + \frac{1}{2}z^{-1}\right)}, \end{aligned} \tag{8}$$

with ROC of $\frac{1}{3} < |z| < \frac{1}{2}$.

Recall another very important z -transform pair:

$$\frac{1}{1 - az^{-1}} \xleftrightarrow{z} \begin{cases} a^n u[n], & |z| > |a| \\ -a^n u[-n - 1], & |z| < |a|. \end{cases} \tag{9}$$

In Eqn (8), the first term corresponds to the right-sided signal pair above and the second pair to the left-sided signal. Since z -transform is linear, we have:

$$x[n] = 2 \left(\frac{1}{3}\right)^n u[n] - 3 \left(-\frac{1}{2}\right)^n u[-n - 1].$$

Problem 6

Given:

$$\begin{aligned}
y[n] &= x_1[-n-2] * x_2[n+4] \\
x_1[n] &= \left(-\frac{1}{2}\right) u[n] \\
x_2[n] &= \left(\frac{1}{4}\right) u[n].
\end{aligned}$$

In this problem, we would like to use Eqn(9) and two more basic properties of z -transform. First, we can find $X_1(z)$ and $X_2(z)$, z -transforms of $x_1[n]$ and $x_2[n]$ respectively:

$$\begin{aligned}
X_1(z) &= \frac{1}{1 + \frac{1}{2}z^{-1}}, \quad |z| > \left|-\frac{1}{2}\right| = \frac{1}{2} \\
X_2(z) &= \frac{1}{1 - \frac{1}{4}z^{-1}}, \quad |z| > \frac{1}{4}.
\end{aligned}$$

$x_2[n+4]$ is nothing but $x_2[n] * \delta[n+4]$. Convolution in time domain corresponds to multiplication in z -transform domain; thus

$$x_2[n+4] \xleftrightarrow{z} X_2(z)z^4 = \frac{z^4}{1 - \frac{1}{4}z^{-1}}, \quad |z| > \frac{1}{4}.$$

Similarly, we want to first compute z -transform of $x_1[n-2]$:

$$x_1[n-2] \xleftrightarrow{z} X_1(z)z^{-2} = \frac{z^{-2}}{1 + \frac{1}{2}z^{-1}}, \quad |z| > \frac{1}{2}.$$

Note that $x_1[-n-2]$ is the time reversed version of $x_1[n-2]$. Thus, using the time reversal property of z -transform, we get:

$$\begin{aligned}
x_1[-n-2] &\xleftrightarrow{z} X_1\left(\frac{1}{z}\right) \left(\frac{1}{z}\right)^{-2} = \frac{z^2}{1 + \frac{1}{2}z}, \quad \left|\frac{1}{z}\right| > \frac{1}{2} \\
\therefore x_1[-n-2] &\xleftrightarrow{z} \frac{z}{z^{-1} + \frac{1}{2}}, \quad |z| < 2.
\end{aligned}$$

Combining together,

$$\begin{aligned}
Y(z) &= \frac{z}{z^{-1} + \frac{1}{2}} \cdot \frac{z^4}{1 - \frac{1}{4}z^{-1}}, \quad \frac{1}{4} < |z| \cap |z| < 2 \\
\therefore Y(z) &= 2 \frac{z^5}{(1 + 2z^{-1})(1 - \frac{1}{4}z^{-1})}, \quad \frac{1}{4} < |z| < 2.
\end{aligned}$$