

**6.003: Signals and Systems—Fall 2003**

PROBLEM SET 4 SOLUTION

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**Home Study Exercise - O&W 3.63**

For an LTI system whose frequency response is:

$$H(j\omega) = \begin{cases} 1, & |\omega| \leq W \\ 0, & |\omega| > W. \end{cases}$$

and which has a continuous-time periodic input signal  $x(t)$  with the following Fourier series representation:

$$x(t) = \sum_{k=-\infty}^{\infty} \alpha^{|k|} e^{jk(\pi/4)t}, \text{ where } \alpha \text{ is a real number between 0 and 1}$$

How large must  $W$  be in order for the output of the system to have at least 90% of the average energy per period of  $x(t)$ ?

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Basically,  $H(j\omega)$  is an ideal Low Pass Filter (LPF) and we need to find how wide it needs to be, in order to pass at least 90% of its input's average energy per period (i.e. average power).

First, let's rewrite the condition above relating the average powers of the input and output, with Fourier series coefficients  $a_k$  and  $b_k$ , respectively:

$$P_x = \sum_{k=-\infty}^{\infty} |a_k|^2, \quad P_y = \sum_{k=-\infty}^{\infty} |b_k|^2$$

The required condition, then, would be:

$$P_y \geq RP_x \Rightarrow \sum_{k=-\infty}^{\infty} |b_k|^2 \geq R \sum_{k=-\infty}^{\infty} |a_k|^2, \text{ where } R=0.9 \quad (*)$$

Then, let's calculate the Fourier series coefficients of the output,  $b_k$ :

$$\because b_k = a_k H(jk\omega_0) \Rightarrow b_k = \begin{cases} a_k, & |k\omega_0| \leq W \\ 0, & |k\omega_0| > W \end{cases} = \begin{cases} a_k, & |k| \leq W/\omega_0 \\ 0, & |k| > W/\omega_0 \end{cases}$$

And finally, we plug the expressions of  $a_k$  and  $b_k$  in the required condition and simplify. By matching the the expression of  $x(t)$  with the synthesis equation, we can conclude that  $\omega_0 = \frac{\pi}{4}$  and  $a_k = \alpha^{|k|}$

$$\begin{aligned} P_x &= \sum_{k=-\infty}^{\infty} |a_k|^2 = \sum_{k=-\infty}^{\infty} |\alpha^{|k|}|^2 = 2 \sum_{k=0}^{\infty} \alpha^{2k} - 1 \quad (\because \alpha \text{ is real}) \\ &= \frac{2}{1 - \alpha^2} - 1 \quad (\because 0 < |\alpha| < 1) \end{aligned}$$

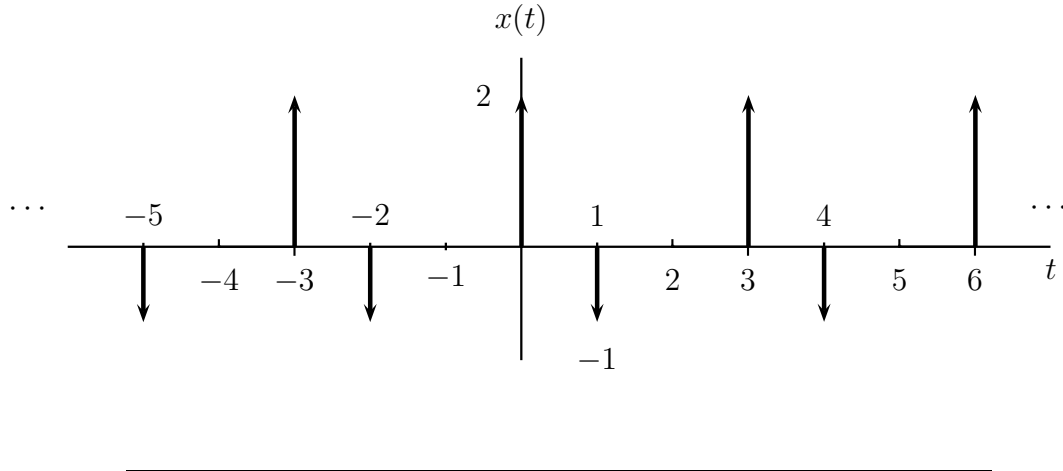
$$\begin{aligned} P_y &= \sum_{k=-\infty}^{\infty} |b_k|^2 = \sum_{k=-N}^N |a_k|^2 \quad , \text{ where } N \text{ is the largest integer, such that } N \leq W/\omega_0 \\ &= \sum_{k=-N}^N |\alpha^{|k|}|^2 = 2 \sum_{k=0}^N \alpha^{2k} - 1 \quad (\because \alpha \text{ is real}) \\ &= 2 \frac{1 - (\alpha^2)^{N+1}}{1 - \alpha^2} - 1 \quad \left( \because \sum_{n=0}^M \beta^n = \frac{1 - \beta^{M+1}}{1 - \beta}, \text{ for any complex } \beta \neq 0 \right) \end{aligned}$$

Plugging in (\*):

$$\begin{aligned} 2 \frac{1 - (\alpha^2)^{N+1}}{1 - \alpha^2} - 1 &\geq R \left( \frac{2}{1 - \alpha^2} - 1 \right) \\ \frac{2 - 2\alpha^{2N+2}}{1 - \alpha^2} &\geq \left( \frac{2R}{1 - \alpha^2} - R + 1 \right) \\ \frac{2 - 2\alpha^{2N+2}}{1 - \alpha^2} &\geq \frac{2R + (1 - R)(1 - \alpha^2)}{1 - \alpha^2} \\ 2 - 2\alpha^{2N+2} &\geq R + 1 - (1 - R)\alpha^2 \quad (\because 1 - \alpha^2 > 0) \\ 2 - 2\alpha^{2N+2} &\geq 1.9 - 0.1\alpha^2 \quad (\text{plugging in } R=0.9) \\ \alpha^{2N+2} &\leq 0.05 + 0.05\alpha^2 \quad (\text{simplifying a bit}) \\ (2N + 2) \log(\alpha) &\leq \log(0.05 + 0.05\alpha^2) \\ N + 1 &\geq \frac{\log(0.05 + 0.05\alpha^2)}{2 \log(\alpha)} \quad (\text{recall that } \alpha < 1 \rightarrow \log(\alpha) < 0) \\ N &\geq \frac{\log(0.05 + 0.05\alpha^2)}{2 \log(\alpha)} - 1 \end{aligned}$$

After choosing an integer  $N$  that satisfies the inequality above,  $W$  can be chosen such that  $W \geq N\omega_0$ .

**Problem 1** Consider the LTI system with impulse response given in O&W 3.34. Find the Fourier series representation of the output  $y(t)$  for the following input.



From O & W 3.34, the impulse response of the LTI system is:

$$h(t) = e^{-4|t|}.$$

From the figure above, we can see that  $x(t)$  has a period  $T = 3 \rightarrow \omega_0 = \frac{2\pi}{3}$ .

First, we calculate the frequency response:

$$\begin{aligned} H(j\omega) &= \int_{-\infty}^{\infty} h(t)e^{-j\omega t} dt = \int_{-\infty}^{\infty} e^{-4|t|} e^{-j\omega t} dt \\ &= \int_{-\infty}^0 e^{-4(-t)} e^{-j\omega t} dt + \int_0^{\infty} e^{-4(t)} e^{-j\omega t} dt \\ &= \int_{-\infty}^0 e^{(4-j\omega)t} dt + \int_0^{\infty} e^{(-4-j\omega)t} dt \\ &= \frac{1}{4-j\omega} e^{(4-j\omega)t} \Big|_{-\infty}^0 + \frac{1}{-4-j\omega} e^{(-4-j\omega)t} \Big|_0^{\infty} \\ &= \frac{1}{4-j\omega} (1-0) + \frac{1}{-4-j\omega} (0-1) \quad (\text{remember that } e^{-\infty+ja} = 0 \text{ for any real } a) \\ &= \frac{1}{4-j\omega} + \frac{1}{4+j\omega} = \frac{4+j\omega+4-j\omega}{16+\omega^2} \\ H(j\omega) &= \frac{8}{16+\omega^2}. \end{aligned}$$

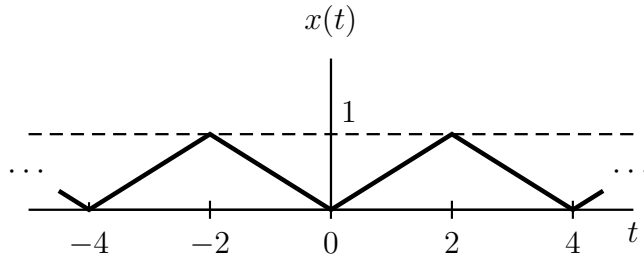
Next, we find the Fourier series coefficients of  $x(t)$ ,  $a_k$ :

$$\begin{aligned} a_k &= \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt = \frac{1}{3} \int_{-1}^2 [2\delta(t) - \delta(t-1)] e^{-jk\omega_0 t} dt \\ &= \frac{1}{3} (2e^{-jk\omega_0(0)} - e^{-jk\omega_0(1)}) \\ &= \frac{2}{3} - \frac{1}{3} e^{-jk\omega_0}, \text{ for all } k. \end{aligned}$$

Now we are ready to find  $b_k$ , the Fourier series representation of the output  $y(t)$ :

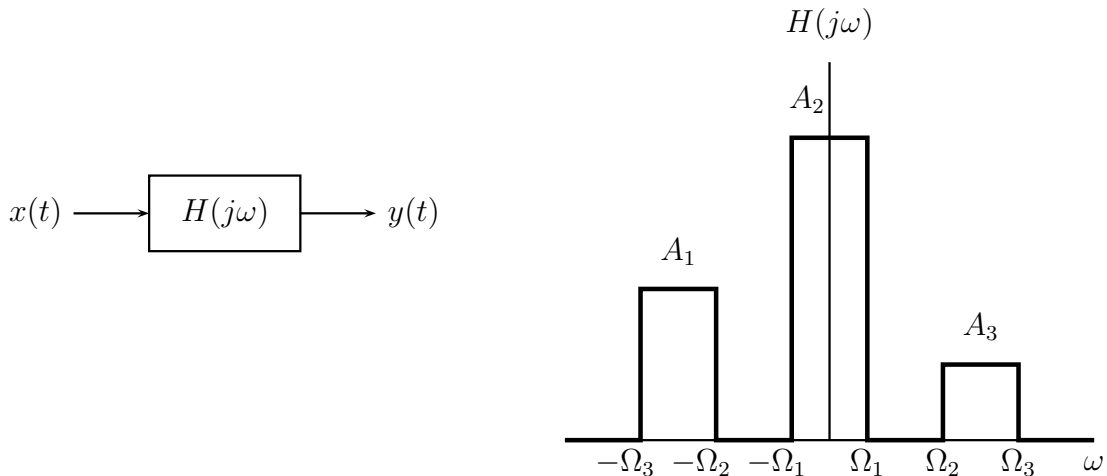
$$\begin{aligned} b_k &= a_k H(jk\omega_0) \quad (\text{O \& W, Section 3.8, p.226, and specifically eq.(3.124) }) \\ &= \left( \frac{2}{3} - \frac{1}{3} e^{-jk\omega_0} \right) \left( \frac{8}{16 + (k\omega_0)^2} \right) \\ &= \left( \frac{8}{3} \right) \frac{2 - e^{-jk\frac{2\pi}{3}}}{16 + \left(k\frac{2\pi}{3}\right)^2}, \text{ for all } k. \end{aligned}$$

**Problem 2** The periodic triangular wave shown below has Fourier series coefficients  $a_k$ .



$$a_k = \begin{cases} 2 \frac{\sin(k\pi/2)}{j(k\pi)^2} e^{-jk\pi/2}, & k \neq 0 \\ \frac{1}{2}, & k = 0. \end{cases}$$

Consider the LTI system with frequency response  $H(j\omega)$  depicted below:



Determine values of  $A_1$ ,  $A_2$ ,  $A_3$ ,  $\Omega_1$ ,  $\Omega_2$ , and  $\Omega_3$  of the LTI filter  $H(j\omega)$  such that

$$y(t) = 1 - \cos\left(\frac{3\pi}{2}t\right).$$

At the beginning, it is worth noting that the output  $y(t)$  contains only a DC component and a single sinusoid with a frequency of  $\frac{3\pi}{2}$ .  $H(j\omega)$  is a linear system so the output will only have frequency components that exist in the input. Knowing that the input  $x(t)$  has a DC component and a fundamental frequency of  $\omega_0 = \frac{\pi}{2}$ , let's dissect  $y(t)$  into a DC component

and complex exponentials with a fundamental frequency of  $\omega_0 = \frac{\pi}{2}$ .

$$\begin{aligned}
y(t) &= 1 - \cos\left(\frac{3\pi}{2}t\right) = 1 - \frac{e^{j\frac{3\pi}{2}t} - \frac{1}{2}e^{-j\frac{3\pi}{2}t}}{2} \\
&= 1 - \frac{1}{2}e^{j(3)\frac{\pi}{2}t} - \frac{1}{2}e^{j(-3)\frac{\pi}{2}t} \\
&= 1 - \frac{1}{2}e^{j(3)\omega_0 t} - \frac{1}{2}e^{j(-3)\omega_0 t} = \sum_k b_k e^{jk\omega_0 t} \\
&\Rightarrow b_k \begin{cases} 1, & k = 0 \\ -\frac{1}{2}, & k = \pm 3 \\ 0, & \text{otherwise} \end{cases}
\end{aligned}$$

$$\because b_k = a_k H(jk\omega_0) \Rightarrow H(jk\omega_0) = \frac{b_k}{a_k}, \text{ for } a_k \neq 0.$$

$y(t)$  has only three non-zero components, therefore  $H(j\omega)$  has a non-zero value at those three components, corresponding to  $A_1, A_2$  and  $A_3$  as follows:

$$\begin{aligned}
H(j(0)\omega_0) &= \frac{b_0}{a_0} = \frac{1}{1/2} = 2 = A_2. \\
H(j(3)\omega_0) &= \frac{b_3}{a_3} = -\frac{1}{2} \div 2 \frac{\sin(3 \cdot \pi/2)}{j(3 \cdot \pi)^2} e^{-j(3)\pi/2} \\
&= -\frac{j(9)\pi^2}{4 \sin(3\pi/2)} e^{j\frac{3\pi}{2}} \\
&= -\frac{j9\pi^2}{4(-1)}(-j) = \frac{9}{4}\pi^2 = A_3. \\
H(j(-3)\omega_0) &= \frac{b_{-3}}{a_{-3}} = -\frac{1}{2} \div 2 \frac{\sin(-3 \cdot \pi/2)}{j(-3 \cdot \pi)^2} e^{-j(-3)\pi/2} \\
&= -\frac{j(9)\pi^2}{4 \sin(-3\pi/2)} e^{j\frac{-3\pi}{2}} \\
&= -\frac{j9\pi^2}{4(1)}(j) = \frac{9}{4}\pi^2 = A_1.
\end{aligned}$$

$H(j\omega)$  also needs to eliminate the other frequency components of  $x(t)$  which do not exist in the output  $y(t)$ .

$$\rightarrow H(jk\omega_0) = 0, \quad \text{for } k \neq 0, \pm 3$$

To meet the above conditions, the cutoff frequencies  $\Omega_{1,2,3}$  must be chosen to pass the desired components and reject the undesired components. The following inequalities meet that

requirement:

$$0 < \Omega_1 < (1)\omega_0, \quad (2)\omega_0 < \Omega_2 < (3)\omega_0, \quad (3)\omega_0 < \Omega_3 < (4)\omega_0 \quad (2.1)$$

Choosing any cutoff frequencies which satisfy (2.1) would be sufficient. Let's say, for practicality sake, that we want to choose the cutoff frequencies in the midpoints between the desired and undesired frequency components. This gives us the following specific values:

$$\begin{aligned} \Omega_1 &= \frac{(0+1)\omega_0}{2}, \quad \Omega_2 = \frac{(2+3)\omega_0}{2}, \quad \Omega_3 = \frac{(3+4)\omega_0}{2}, \quad \text{where } \omega_0 = \frac{\pi}{2} \\ &\rightarrow \Omega_1 = \frac{\pi}{4}, \quad \Omega_2 = \frac{5\pi}{4}, \quad \Omega_3 = \frac{7\pi}{2}. \end{aligned}$$

**Problem 3** Consider a causal discrete-time LTI system whose input  $x[n]$  and output  $y[n]$  are related by the following difference equation:

$$y[n] - \frac{1}{4}y[n-1] = x[n] + 2x[n-4]$$

Find the Fourier series representation of the output  $y[n]$  when the input is

$$x[n] = 2 + \sin(\pi n/4) - 2 \cos(\pi n/2).$$

First, let's find the frequency response of the system from the difference equation by injecting an input,  $x[n]$ , that is an eigenfunction of the LTI system:

$$x[n] = e^{j\omega n} \rightarrow y[n] = H(e^{j\omega}) e^{j\omega n}$$

$H(e^{j\omega})$  is the frequency response characterizing the system or the eigenvalue of the system. By substituting  $x[n]$  and  $y[n]$  in the difference equation:

$$\begin{aligned} y[n] - \frac{1}{4}y[n-1] &= x[n] + 2x[n-4] \\ H(e^{j\omega}) e^{j\omega n} - \frac{1}{4} \cdot H(e^{j\omega}) e^{j\omega(n-1)} &= e^{j\omega n} + 2 \cdot e^{j\omega(n-4)} \\ H(e^{j\omega}) e^{j\omega n} - \frac{1}{4} \cdot H(e^{j\omega}) e^{j\omega n} e^{j\omega(-1)} &= e^{j\omega n} + 2 \cdot e^{j\omega n} e^{j\omega(-4)} \\ H(e^{j\omega}) e^{j\omega n} \left[ 1 - \frac{1}{4} e^{-j\omega} \right] &= e^{j\omega n} [1 + 2 e^{-j\omega 4}] \\ \rightarrow H(e^{j\omega}) &= \frac{1 + 2 e^{-j\omega 4}}{1 - \frac{1}{4} e^{-j\omega}}. \end{aligned}$$

Then, we find the Fourier series coefficients,  $a_k$ , of the given input, possibly by dissecting the input expression into a summation of complex exponentials:

$$\begin{aligned} x[n] &= 2 + \sin(\pi n/4) - 2 \cos(\pi n/2) = 2 + \sin(\omega_0 n) - 2 \cos(2\omega_0 n), \\ &\text{where } \omega_0 = \frac{\pi}{4} \text{ is the Greatest Common Factor for the sinusoids frequencies} \\ &= 2 e^{j(0)\omega_0 n} + \frac{e^{j\omega_0 n} - e^{-j\omega_0 n}}{2j} - 2 \cdot \frac{e^{j2\omega_0 n} + e^{-j2\omega_0 n}}{2} \\ &= 2 e^{j(0)\omega_0 n} + \frac{1}{2j} e^{j(1)\omega_0 n} - \frac{1}{2j} e^{j(-1)\omega_0 n} - e^{j(2)\omega_0 n} - e^{j(-2)\omega_0 n} \end{aligned}$$

$\therefore N = 8 \rightarrow a_k$  has only eight distinct values and is periodic with a period of  $N = 8$ .

$$\Rightarrow a_k = \begin{cases} -1, & k = -2 \\ -\frac{1}{2j}, & k = -1 \\ 2, & k = 0 \\ \frac{1}{2j}, & k = 1 \\ -1, & k = 2 \\ 0, & k = 3, 4, 5 \end{cases}$$

And finally, we find the Fourier series coefficients,  $b_k$ , of the output  $y[n]$ :

$$\begin{aligned} b_k &= a_k H(e^{jk\omega_0}) = a_k \cdot \frac{1 + 2e^{-jk\omega_0}}{1 - \frac{1}{4}e^{-jk\omega_0}} = a_k \cdot \frac{1 + 2e^{-jk\frac{\pi}{4}}}{1 - \frac{1}{4}e^{-jk\frac{\pi}{4}}} \\ &= a_k \cdot \frac{1 + 2e^{-jk\pi}}{1 - \frac{1}{4}e^{-jk\frac{\pi}{4}}} \end{aligned}$$

We have only few non-zero coefficients, so we can go ahead and evaluate them. In doing so, it is sometimes useful while computing the value of the complex exponential, to visualize the the complex vector  $e^{jk\theta}$  going around the unit circle. As the integer  $k$  increases by one, the complex vector's angle increases by an angle of  $\theta$ .

$$\begin{aligned} b_0 &= a_0 H(e^{j(0)\omega_0}) = (2) \frac{1 + 2e^{-j(0)\pi}}{1 - \frac{1}{4}e^{-j(0)\frac{\pi}{4}}} = (2) \frac{1 + 2}{1 - \frac{1}{4}} = \frac{6}{\frac{3}{4}} \\ &= 8 \\ b_1 &= a_1 H(e^{j(1)\omega_0}) = \left(\frac{1}{2j}\right) \frac{1 + 2e^{-j(1)\pi}}{1 - \frac{1}{4}e^{-j(1)\frac{\pi}{4}}} = \left(\frac{1}{2j}\right) \frac{1 + 2(-1)}{1 - \frac{1}{4}\left(\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right)} \\ &= 0.1247 + j 0.5806 \\ b_{-1} &= a_{-1} H(e^{j(-1)\omega_0}) = \left(\frac{-1}{2j}\right) \frac{1 + 2e^{-j(-1)\pi}}{1 - \frac{1}{4}e^{-j(-1)\frac{\pi}{4}}} = \left(\frac{-1}{2j}\right) \frac{1 + 2(-1)}{1 - \frac{1}{4}\left(\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\right)} \\ &= 0.1247 - j 0.5806 \\ b_2 &= a_2 H(e^{j(2)\omega_0}) = (-1) \frac{1 + 2e^{-j(2)\pi}}{1 - \frac{1}{4}e^{-j(2)\frac{\pi}{4}}} = (-1) \frac{1 + 2(1)}{1 - \frac{1}{4}(-j)} \\ &= -2.8235 + j 0.7059 \\ b_{-2} &= a_{-2} H(e^{j(-2)\omega_0}) = (-1) \frac{1 + 2e^{-j(-2)\pi}}{1 - \frac{1}{4}e^{-j(-2)\frac{\pi}{4}}} = (-1) \frac{1 + 2(1)}{1 - \frac{1}{4}(j)} \\ &= -2.8235 - j 0.7059 \\ b_{3,4,5} &= 0. \end{aligned}$$

As a double-check for our answer, notice that  $b_{-k} = b_k^*$  which indicates that the output  $y[n]$  is real. We expect this because the input is real and because the LTI system applies a real operation, i.e. the difference equation. We also expect this from a mathematical point of

view because the input is real, i.e.  $a_{-k} = a_k^*$ , and  $H(e^{-j\omega}) = H^*(e^{j\omega})$ , from the calculated expression.

**Problem 4** Specify the frequency response of a discrete-time LTI system so that if the input is

$$x[n] = 2 + \cos(\pi n) - \sin(\pi n/2) + 2 \cos(\pi n/4 + \pi/4)$$

then the output is

$$y[n] = 4 - 2 \sin(\pi n) + 2 \cos(\pi n/4).$$

It will be straight forward to determine the frequency response of the system, once we expand the expressions for  $x[n]$  and  $y[n]$  into their complex exponential components:

$$\begin{aligned} x[n] &= 2 + \cos(\pi n) - \sin(\pi n/2) + 2 \cos(\pi n/4 + \pi/4) \\ &= 2 + \left( \frac{1}{2} e^{j\pi n} + \frac{1}{2} e^{-j\pi n} \right) - \left( \frac{1}{2j} e^{j\pi n/2} - \frac{1}{2j} e^{-j\pi n/2} \right) + 2 \left( \frac{1}{2} e^{j(\pi n/4 + \pi/4)} + \frac{1}{2} e^{-j(\pi n/4 + \pi/4)} \right) \\ &= 2e^{j0} + \frac{1}{2} e^{j\pi n} + \frac{1}{2} e^{-j\pi n} - \frac{1}{2j} e^{j\pi n/2} + \frac{1}{2j} e^{-j\pi n/2} + e^{j\pi n/4} e^{j\pi/4} + e^{-j\pi n/4} e^{-j\pi/4} \end{aligned} \quad (1)$$

$$\begin{aligned} y[n] &= 4 - 2 \sin(\pi n) + 2 \cos(\pi n/4) \\ &= 4 - 2 \left( \frac{1}{2j} e^{j\pi n} - \frac{1}{2j} e^{-j\pi n} \right) + 2 \left( \frac{1}{2} e^{j\pi n/4} + \frac{1}{2} e^{-j\pi n/4} \right) \\ &= 4e^{j0} - \frac{1}{j} e^{j\pi n} + \frac{1}{j} e^{-j\pi n} + e^{j\pi n/4} + e^{-j\pi n/4} \end{aligned} \quad (2)$$

Now we can find  $H(e^{j\omega})$  using the following two approaches:

- (I) Consider equation (1) to be the summation of the eigenfunctions  $e^{j\omega n}$  that comprise the input,  $x[n]$ . By superposition, the output  $y[n]$  will contain the same eigenfunctions multiplied by the system's eigenvalues, i.e  $e^{j\omega n} H(e^{j\omega})$  (refer to O & W, Section 3.2, p.183).

$$\begin{aligned} x[n] &= 2e^{j0} + \frac{1}{2} e^{j\pi n} + \frac{1}{2} e^{-j\pi n} - \frac{1}{2j} e^{j\pi n/2} + \frac{1}{2j} e^{-j\pi n/2} + e^{j\pi/4} e^{j\pi n/4} + e^{-j\pi/4} e^{-j\pi n/4} \\ y[n] &= 4e^{j0} - \frac{1}{j} e^{j\pi n} + \frac{1}{j} e^{-j\pi n} + (0)e^{j\pi n/2} + (0)e^{-j\pi n/2} + e^{j\pi n/4} + e^{-j\pi n/4} \end{aligned}$$

By matching the eigenfunctions in the input and the output, we can find the values of

the frequency response:

$$\rightarrow H(e^{j\omega}) = \begin{cases} -j2, & \omega = -\pi \\ e^{j\pi/4}, & \omega = -\frac{\pi}{4} \\ 2, & \omega = 0 \\ e^{-j\pi/4}, & \omega = \frac{\pi}{4} \\ j2, & \omega = \pi \\ 0, & \omega = \pm\frac{\pi}{2} \\ ?, & \text{otherwise} \end{cases}$$

The question mark in the above expression indicates that we don't have enough information to determine the system's frequency response at those frequencies. This is simply because the input that excited the system did not contain those frequencies.

- (II) Consider equations (1) and (2) to be the synthesis equations for  $x[n]$  and  $y[n]$ , respectively, and assume the fundamental frequency,  $\omega_0$ , to be the Greatest Common Factor for the sinusoids frequencies, i.e.  $\omega_0 = \frac{\pi}{4}$  and  $N = 8$ .

$$\begin{aligned} x[n] &= 2e^{j0} + \frac{1}{2}e^{j\pi n} + \frac{1}{2}e^{-j\pi n} - \frac{1}{2j}e^{j\pi n/2} + \frac{1}{2j}e^{-j\pi n/2} + e^{j\pi n/4}e^{j\pi/4} + e^{-j\pi n/4}e^{-j\pi/4} \\ &= (2)e^{j(0)\omega_0 n} + \left(\frac{1}{2}\right)e^{j(4)\omega_0 n} + \left(\frac{1}{2}\right)e^{j(-4)\omega_0 n} + \left(\frac{-1}{2j}\right)e^{j(2)\omega_0 n} + \left(\frac{1}{2j}\right)e^{j(-2)\omega_0 n} \\ &\quad + (e^{j\pi/4})e^{j(1)\omega_0 n} + (e^{-j\pi/4})e^{j(-1)\omega_0 n} \\ y[n] &= 4e^{j0} - \frac{1}{j}e^{j\pi n} + \frac{1}{j}e^{-j\pi n} + e^{j\pi n/4} + e^{-j\pi n/4} \\ &= (4)e^{j(0)\omega_0 n} + (j)e^{j(4)\omega_0 n} + (-j)e^{j(-4)\omega_0 n} + 0 + 0 + (1)e^{j(1)\omega_0 n} + (1)e^{j(-1)\omega_0 n}, \end{aligned}$$

where the integer between parenthesis in the exponential corresponds to the index  $k$  and the number between parenthesis in front of the exponential is the corresponding  $a_k$  and  $b_k$  for  $x[n]$  and  $y[n]$ , respectively.

$$\therefore H(e^{jk\omega_0}) = \frac{b_k}{a_k} \rightarrow H(e^{jk\frac{\pi}{4}}) = \begin{cases} -j2, & k = -4 \\ e^{j\pi/4}, & k = -1 \\ 2, & k = 0 \\ e^{-j\pi/4}, & k = 1 \\ j2, & k = 4 \\ 0, & k = \pm 2 \\ ?, & \text{otherwise} \end{cases}$$

**Problem 5** Compute the Fourier transform of each of the following signals:

(a)  $x(t) = e^{-|t|} \cos 2t$

Trying to compute the Fourier transform of  $x(t)$  using the analysis equation (O & W, p.288) might require going through a lengthy integration. Instead we will use the Fourier transform properties.

Let  $x(t) = e^{-|t|} \cos 2t = s(t)p(t)$ , where  $s(t) = e^{-|t|}$  and  $p(t) = \cos 2t$ .

From the Multiplication property (O & W, Section 4.5, p.322) we have:

$$X(j\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(j\theta)P(j(\omega - \theta))d\theta = \frac{1}{2\pi} S(j\omega) * P(j\omega)$$

Now, we need to find the Fourier transforms of  $s(t)$  and  $p(t)$  and plug them in the expression above.

From Example 4.2 (O & W, p.291):  $e^{-a|t|} \xleftrightarrow{\mathcal{F}} \frac{2a}{a^2 + \omega^2}$

$$\therefore s(t) = e^{-|t|} \xleftrightarrow{\mathcal{F}} S(j\omega) = \frac{2}{1 + \omega^2}$$

From Table 4.2 (O & W, p.329):  $\cos \omega_0 t \xleftrightarrow{\mathcal{F}} \pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$

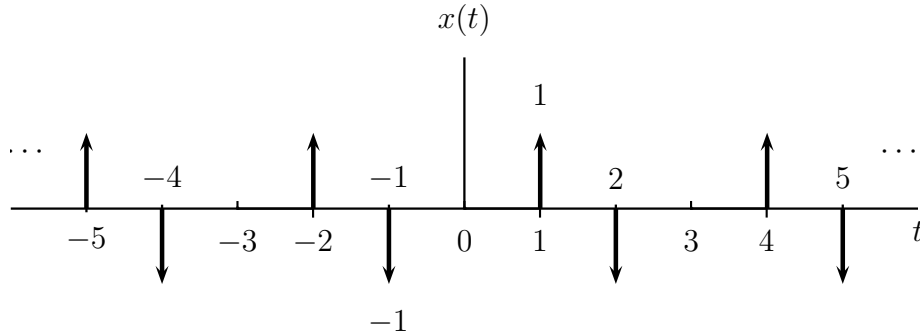
$$\therefore p(t) = \cos 2t \xleftrightarrow{\mathcal{F}} P(j\omega) = \pi[\delta(\omega - 2) + \delta(\omega + 2)]$$

$$\begin{aligned} X(j\omega) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} S(j\theta)P(j(\omega - \theta))d\theta \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2}{1 + \theta^2} \pi[\delta(\omega - 2 - \theta) + \delta(\omega + 2 - \theta)]d\theta \\ &= \frac{1}{1 + (\omega - 2)^2} + \frac{1}{1 + (\omega + 2)^2} \quad \left( \text{recalling that } \int_{-\infty}^{\infty} g(t)\delta(t - t_0)dt = g(t_0) \right) \end{aligned}$$

Let's double check our answer, using the analysis equation, and considering that the cosine function can be expressed as a sum of complex exponentials.

$$\begin{aligned}
X(j\omega) &= \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} e^{-|t|} \cos 2t e^{-j\omega t} dt \\
&= \int_{-\infty}^0 e^t \cos 2t e^{-j\omega t} dt + \int_0^{\infty} e^{-t} \cos 2t e^{-j\omega t} dt \\
&= \int_{-\infty}^0 e^t \left[ \frac{1}{2} e^{j2t} + \frac{1}{2} e^{-j2t} \right] e^{-j\omega t} dt + \int_0^{\infty} e^{-t} \left[ \frac{1}{2} e^{j2t} + \frac{1}{2} e^{-j2t} \right] e^{-j\omega t} dt \\
&= \frac{1}{2} \int_{-\infty}^0 e^{t(1+j2-j\omega)} + e^{t(1-j2-j\omega)} dt + \frac{1}{2} \int_0^{\infty} e^{t(-1+j2-j\omega)} + e^{t(-1-j2-j\omega)} dt \\
&= \frac{1}{2} \left( \frac{1}{1+j2-j\omega} + \frac{1}{1-j2-j\omega} \right) + \frac{1}{2} \left( \frac{-1}{-1+j2-j\omega} + \frac{-1}{-1-j2-j\omega} \right) \\
&= \frac{1}{2} \left( \frac{1}{1+j(2-\omega)} + \frac{1}{1-j(2-\omega)} + \frac{1}{1-j(2+\omega)} + \frac{1}{1+j(2+\omega)} \right) \\
&= \frac{1}{1+(2-\omega)^2} + \frac{1}{1+(2+\omega)^2} \quad , \text{ which is the same answer.}
\end{aligned}$$

(b) The signal  $x(t)$  depicted below:



Note that the signal  $x(t)$  is composed of two time-shifted impulse trains, so we will use the following Fourier transform property and basic Fourier transform pair to find  $X(j\omega)$  (see Tables 4.1 and 4.2, O&W, p.328-29):

$$x(t - t_o) \xleftrightarrow{\mathcal{F}} e^{-j\omega t_o} X(j\omega) \quad , \quad \sum_{n=-\infty}^{\infty} \delta(t - nT) \xleftrightarrow{\mathcal{F}} \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$$

$$\begin{aligned}
x(t) &= x_1(t) + x_2(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT - 1) - \sum_{n=-\infty}^{\infty} \delta(t - nT + 1) \quad , \text{ where } T=3 \\
\rightarrow X(j\omega) &= (e^{-j\omega} - e^{j\omega}) \frac{2\pi}{3} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{3}\right) = \frac{4\pi}{3j} \sin \omega \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{3}\right).
\end{aligned}$$

**Problem 6** Determine the continuous-time signal corresponding to each of the following transforms:

$$(a) X(j\omega) = j[\delta(\omega + 1) - \delta(\omega - 1)] - 3[\delta(\omega - \pi) + \delta(\omega + \pi)]$$

From the Fourier Transform of sinusoids in table 4.2 (O & W, p.3.29)

$$\cos \omega_0 t \xleftrightarrow{\mathcal{F}} \pi [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)] \quad , \quad \sin \omega_0 t \xleftrightarrow{\mathcal{F}} \frac{\pi}{j} [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$$

$$\begin{aligned} X(j\omega) &= j[\delta(\omega + 1) - \delta(\omega - 1)] - 3[\delta(\omega - \pi) + \delta(\omega + \pi)] \\ &= \frac{1}{\pi} \frac{\pi}{j} [\delta(\omega - 1) - \delta(\omega + 1)] - \frac{3}{\pi} \pi [\delta(\omega - \pi) + \delta(\omega + \pi)] \\ &\rightarrow x(t) = \frac{1}{\pi} \sin t - \frac{3}{\pi} \cos \pi t. \end{aligned}$$

$$(b) X(j\omega) = 2 \sin(2\omega - \pi/2)$$

$$\begin{aligned} X(j\omega) &= 2 \sin(2\omega - \pi/2) = 2 \left( \frac{1}{2j} e^{j(2\omega - \frac{\pi}{2})} - \frac{1}{2j} e^{-j(2\omega - \frac{\pi}{2})} \right) \\ &= \frac{1}{j} e^{j2\omega} e^{-j\frac{\pi}{2}} - \frac{1}{j} e^{-j2\omega} e^{j\frac{\pi}{2}} \\ &= \frac{1}{j} e^{j2\omega} (-j) - \frac{1}{j} e^{-j2\omega} (j) = -e^{j2\omega} - e^{-j2\omega} \\ &\rightarrow x(t) = -\delta(t + 2) - \delta(t - 2) \quad , \text{ from table 4.2 (O \& W, p.3.29) .} \end{aligned}$$

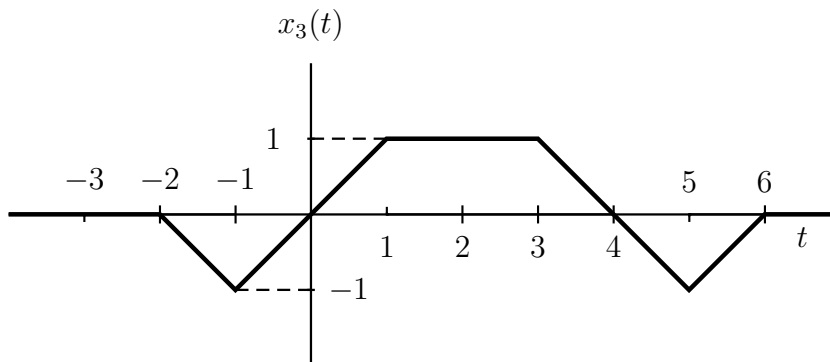
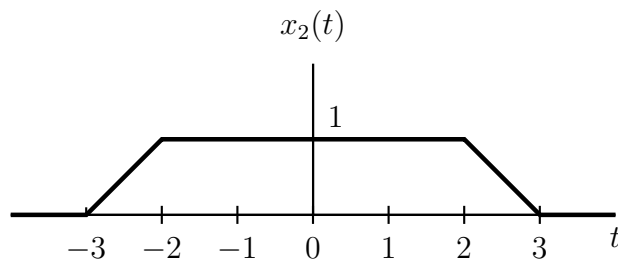
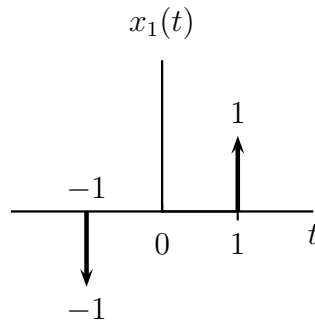
Alternatively we can express  $X(j\omega)$  as:

$$X(j\omega) = 2 \sin(2\omega - \pi/2) = -2 \cos(2\omega)$$

Using the duality property (O & W, Section 4.3.6, p.309) :

$$\begin{aligned} \frac{1}{2} \delta(t - t_o) + \frac{1}{2} \delta(t + t_o) &\xleftrightarrow{\mathcal{F}} \cos \omega t_o \\ \rightarrow x(t) &= -2 \left[ \frac{1}{2} \delta(t - 2) + \frac{1}{2} \delta(t + 2) \right] = -\delta(t - 2) - \delta(t + 2). \end{aligned}$$

**Problem 7** Answer the questions asked in O&W 4.24 (a) for each of the following signals:



Determine which, if any, of the real signals depicted have Fourier transforms that satisfy each of the following conditions:

(1)  $\Re\{X(j\omega)\} = 0$

Before testing this and other conditions, let's review a useful property of the Fourier transform of a real signal  $x(t)$ :

$$\mathcal{E}v\{x(t)\} \xleftrightarrow{\mathcal{F}} \Re\{X(j\omega)\}, \mathcal{O}d\{x(t)\} \xleftrightarrow{\mathcal{F}} j\Im\{X(j\omega)\} \quad (\text{O \& W, Section 4.3.3, p. 303}).$$

Now to test for this condition, note that  $X(j\omega)$  will have no real part only if  $x(t)$  is an odd function, i.e  $x(t)$  has only an odd component  $\rightarrow x(-t) = -x(t)$ .

By inspection, it is easy to see that only  $x_1(t)$  is odd. Therefore this condition is true for  $x_1(t)$  and false for  $x_2(t)$  and  $x_3(t)$ . Note that  $x_3(t)$  can not be described as either odd or even, which means that it has both even and odd components, and hence its Fourier transform would have both real and imaginary parts.

$$(2) \Im\{X(j\omega)\} = 0$$

Similar to the previous condition,  $X(j\omega)$  will have no imaginary part only if  $x(t)$  is an even function, i.e  $x(t)$  has only an even component  $\rightarrow x(-t) = x(t)$ .

By inspection,  $x_2(t)$  is the only even signal, and hence this condition is true for  $x_2(t)$  and false for  $x_1(t)$  and  $x_3(t)$ .

$$(3) \text{ There exists a real } \alpha \text{ such that } e^{j\alpha\omega}X(j\omega) \text{ is real}$$

Again, the only way for a real signal to have a real Fourier transform is if that signal is purely even. The complex exponential that is multiplied by  $X(j\omega)$  hints to time-shifting (see O & W, Section 4.3.2, p. 301). So what this condition tests is whether the signal can be made even by shifting it in time.

For an aperiodic signal this condition can be true only if the signal has both even and odd components. The reason is that for an aperiodic signal to be even or odd, it must have symmetry. Time-shifting such a signal will destroy that symmetry so that the signal can not be described as even or odd afterward. On the other hand, this is not true for periodic signals. For example the cosine and sine functions are even and odd, respectively, and one can be converted to the other by time-shifting (for example by a quarter-period time-shift).

Coming back to the three signals at hand:

$x_1(t)$  is odd and can not be made even by time-shifting, as we explained.

$x_2(t)$  is already even  $\rightarrow$  it satisfies the condition, with  $\alpha = 0$ .

$x_3(t)$  is even symmetric about  $t = 2$   $\rightarrow$  it satisfies the condition, with  $\alpha = 2$ .

$$(4) \int_{-\infty}^{\infty} X(j\omega)d\omega = 0$$

Let's manipulate the integral a bit to see that this condition just means that  $x(0) = 0$ :

$$\begin{aligned} \int_{-\infty}^{\infty} X(j\omega)d\omega &= \int_{-\infty}^{\infty} X(j\omega)e^{j\omega(0)}d\omega = 2\pi \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)e^{j\omega(0)}d\omega \right] \\ &= 2\pi \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \right]_{t=0} \\ &= 2\pi x(0) \quad (\text{synthesis equation: O\&W, Section 4.1.1, p. 288}) \end{aligned}$$

By inspecting the values of the signals at  $t = 0$ , we can see that this condition is true for  $x_1(t)$  and  $x_3(t)$  and false for  $x_2(t)$ .

$$(5) \int_{-\infty}^{\infty} \omega X(j\omega) d\omega = 0$$

Similar to the previous condition, let's manipulate the integral a bit:

$$\begin{aligned} \int_{-\infty}^{\infty} \omega X(j\omega) d\omega &= \int_{-\infty}^{\infty} \omega X(j\omega) e^{j\omega(0)} d\omega = \frac{2\pi}{j} \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} j\omega X(j\omega) e^{j\omega(0)} d\omega \right] \\ &= \frac{2\pi}{j} \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} j\omega X(j\omega) e^{j\omega t} d\omega \right]_{t=0} \\ &= \frac{2\pi}{j} \left. \frac{d}{dt} x(t) \right|_{t=0}, \text{ from the Differentiation in Time property} \end{aligned}$$

By checking the condition that  $x'_i(0) = 0$ , we can see that it is true for  $x_1(t)$  and  $x_2(t)$  and false for  $x_3(t)$ .

$$(6) X(j\omega) \text{ is periodic}$$

A possible way to check this condition is using Parseval's Relation for Aperiodic Signals:

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega \quad (\text{O \& W, Section 4.3.7, p. 312})$$

The value computed by each side in the equation above is the energy of the signal.

In general, a real finite signal would have finite energy if the signal were time-limited, i.e. there exists a real  $\alpha > 0$ , such that  $x(t) = 0$ , for  $|t| > \alpha$ . In contrast, a real finite signal that is periodic has finite power and infinite energy.

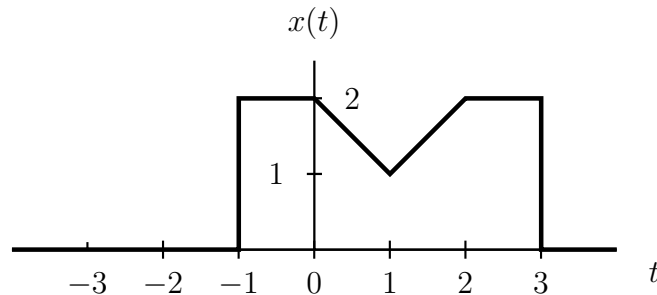
Similarly, a signal with a periodic Fourier transform has an infinite energy. This means, by Parseval's theorem, that a signal that has finite energy will not have a periodic Fourier transform. For a related discussion, see Section 4.1.2, O & W, p.289.

Using this approach, we can see that  $x_2(t)$  and  $x_3(t)$  both have finite energy, i.e.  $\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty$ , and hence do not have a periodic Fourier transform.

An impulse is not finite, so we cannot apply the previous test on  $x_1(t)$ . However, notice that  $x(t)$  has a close resemblance to the Fourier transform of a sine wave (see Table 4.3, O & W, p. 329). By duality, this indicates that the  $X_3(j\omega)$  will be sinusoidal, and hence is periodic.

**Problem 8** O&W 4.25.

Let  $X(j\omega)$  denote the Fourier transform of the signal  $x(t)$  depicted in Figure P4.25.



- (a)  $X(j\omega)$  can be written as  $A(j\omega)e^{j\theta(j\omega)}$  where  $A(j\omega)$  and  $\theta(j\omega)$  are real. Find  $\theta(j\omega)$ .

$$X(j\omega) = A(j\omega)e^{j\theta(j\omega)} = |X(j\omega)|e^{j\angle X(j\omega)}$$

$\because x(t)$  is symmetric about  $t = 1$  (from Figure P4.25, redrawn above).

$\rightarrow$  a signal  $g(t) = x(t + 1)$  is symmetric about  $t=0$

$\rightarrow g(t)$  is even  $\rightarrow G(j\omega)$  is real.

$$\therefore x(t) = g(t - 1) \xrightarrow{\mathcal{F}} X(j\omega) = G(j\omega)e^{-j\omega(1)} = A(j\omega)e^{j\theta(j\omega)}$$

Before going through the last step to find  $\theta(j\omega)$ , let's underline an important subtlety: If we assume that  $A(j\omega) = |X(j\omega)|$  and  $\theta(j\omega) = \angle X(j\omega)$ , then it might be impossible to find  $\theta(j\omega)$  without actually computing  $\angle G(j\omega)$ . However, we are supposed to solve the problem without explicitly evaluating any Fourier Transforms. The reason is that although  $G(j\omega)$  is real, that doesn't mean  $\angle G(j\omega) = 0$ . This is because  $G(j\omega)$  might have a negative value in some range of  $\omega$ . In this case,  $\angle G(j\omega) = \pm\pi$ , because the magnitude, by definition, has to be positive.

Luckily, there is a way out of this dilemma: the only restriction we have is that  $A(j\omega)$  and  $\theta(j\omega)$  be real. If we include the sign of  $X(j\omega)$  in  $A(j\omega)$ , in which case  $A(j\omega)$  is still real but not necessary positive, then we are all set. In this case

$$\because X(j\omega) = G(j\omega)e^{-j\omega(1)} = A(j\omega)e^{j\theta(j\omega)} \text{ and } \because G(j\omega) \text{ is real}$$

$\therefore$  a possible matching of the LHS and the RHS is:

$$A(j\omega) = G(j\omega) \quad \text{and} \quad e^{j\theta(j\omega)} = e^{-j\omega(1)} = e^{-j\omega} \\ \rightarrow \theta(j\omega) = -\omega.$$

- (b) Find  $X(j0)$

$$X(j0) = \int_{-\infty}^{\infty} x(t)e^{-j\omega(0)} dt = \int_{-\infty}^{\infty} x(t) dt = (\text{total area under the curve}) \\ X(j0) = 2[3 - (-1)] - (1)(1) = 7.$$

(c) Find  $\int_{-\infty}^{\infty} X(j\omega) d\omega$ .

$$\because x(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) d\omega \rightarrow \int_{-\infty}^{\infty} X(j\omega) d\omega = 2\pi x(0) = 2\pi(2) = 4\pi.$$

(d) Evaluate  $\int_{-\infty}^{\infty} X(j\omega) \frac{2\sin\omega}{\omega} e^{j2\omega} d\omega$ .

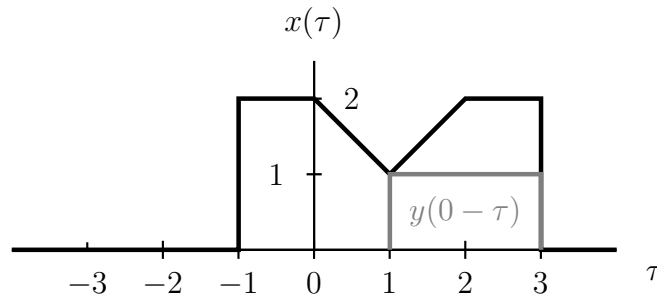
Let  $Y(j\omega) = \frac{2\sin\omega}{\omega} e^{j2\omega}$ , therefore:

$$\begin{aligned} \int_{-\infty}^{\infty} X(j\omega) \frac{2\sin\omega}{\omega} e^{j2\omega} d\omega &= \int_{-\infty}^{\infty} X(j\omega) Y(j\omega) d\omega \\ &= 2\pi \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) Y(j\omega) e^{j\omega(0)} d\omega \right] \\ &= 2\pi \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) Y(j\omega) e^{j\omega t} d\omega \right]_{t=0} \\ &= 2\pi [x(t) * y(t)]_{t=0} \quad (\text{see O \& W, Sec. 4.4, p.314}) \end{aligned}$$

Knowing that  $g(t) = \begin{cases} 0, & |t| < T_1 \\ 1, & |t| > T_1 \end{cases} \xleftrightarrow{\mathcal{F}} \frac{2\sin\omega T_1}{\omega}$  ( From O & W, Table 4.2, p.329 or Example 4.4, p. 293), therefore:

$$y(t) = \begin{cases} 1, & -3 < t < -1 \\ 0, & \text{otherwise} \end{cases} \xleftrightarrow{\mathcal{F}} Y(j\omega) = \frac{2\sin\omega(1)}{\omega} e^{j\omega(2)}$$

$\rightarrow x(t) * y(t)|_{t=0} = \int_1^3 x(\tau) d\tau = 3.5$  (as seen in the figure, below, depicting the convolution)



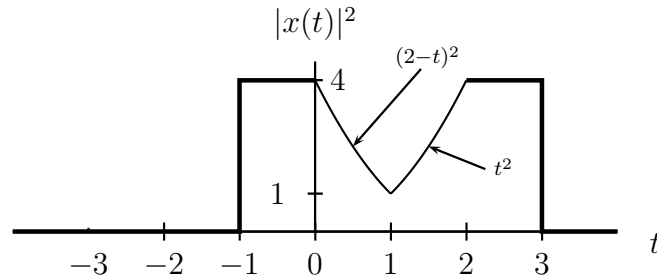
$$\rightarrow \int_{-\infty}^{\infty} X(j\omega) \frac{2\sin\omega}{\omega} e^{j2\omega} d\omega = 2\pi(3.5) = 7\pi.$$

(e) Evaluate  $\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$

From Parseval's theorem:

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega \quad (\text{O \& W, Section 4.3.7, p. 312})$$

$$\begin{aligned} \rightarrow \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega &= 2\pi \int_{-\infty}^{\infty} |x(t)|^2 dt \\ &= 2\pi \left[ \int_{-1}^0 (2)^2 dt + \int_0^1 (2-t)^2 dt + \int_1^2 (t)^2 dt + \int_2^3 (2)^2 dt \right] \\ &= 2\pi \left[ 4 + \left. \frac{(2-t)^3}{-3} \right|_0^1 + \left. \frac{t^3}{3} \right|_1^2 + 4 \right] = 2\pi \left[ 4 - \left( \frac{1}{3} - \frac{8}{3} \right) + \frac{8}{3} - \frac{1}{3} + 4 \right] \\ &= 2\pi \left( \frac{38}{3} \right) = \frac{76\pi}{3}. \end{aligned}$$



Note that a useful Fourier transform property that we have used several times now is the following:

$$2\pi x(0) \xleftrightarrow{\mathcal{F}} \int_{-\infty}^{\infty} X(j\omega) d\omega, \text{ and by duality: } \int_{-\infty}^{\infty} x(t) dt \xleftrightarrow{\mathcal{F}} X(j0).$$

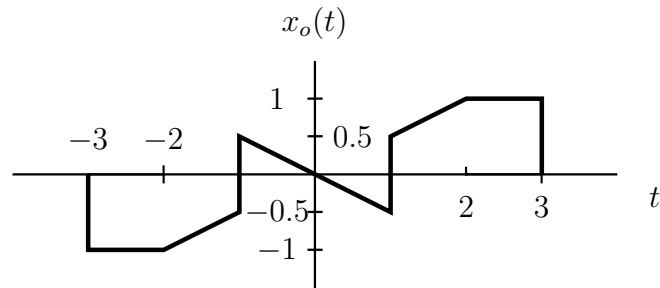
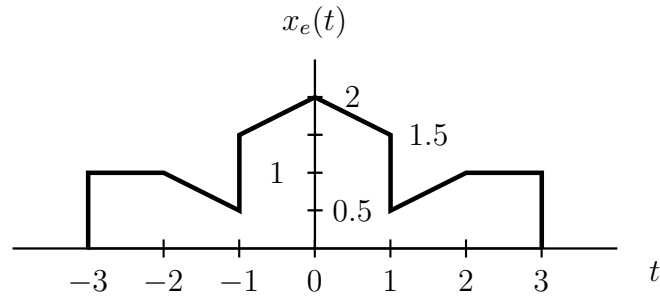
(f) Sketch the inverse Fourier transform of  $\Re\{X(j\omega)\}$ .

The key to answering this question is recalling that the real part of a Fourier transform corresponds to the even part of the signal (as discussed in problem 7):

$$\mathcal{E}v\{x(t)\} \xleftrightarrow{\mathcal{F}} \Re\{X(j\omega)\}, \mathcal{O}d\{x(t)\} \xleftrightarrow{\mathcal{F}} j\Im\{X(j\omega)\} \quad (\text{O \& W, Section 4.3.3, p. 303}).$$

To resolve the even part, we use the following formulae:

$$x_e = \mathcal{E}v\{x(t)\} = \frac{1}{2}[x(t) + x(-t)] \quad , \quad x_o = \mathcal{O}d\{x(t)\} = \frac{1}{2}[x(t) - x(-t)]$$



You might want to double-check that  $x_o(t) + x_e(t) = x(t)$ . Note that the sketch for the odd part of  $x(t)$  is included here for illustration purposes, and was not required in the original problem.

As a last note, one might be tempted to find the inverse Fourier transform of  $\Re\{X(j\omega)\}$  by shifting  $x(t)$  to the left by one unit, and hence making it even symmetric which would have a real Fourier transform. It will be easy to convince yourself of the falsity of that method, if you remember that shifting a signal in time changes its Fourier transform's angle but does not affect the magnitude. This means that the real part of the Fourier transform of a signal changes with the time-shifting of that signal. (Hint:  $Ae^{j\theta} = A \cos \theta + jA \sin \theta$ ).