MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

**6.003: Signals and Systems—Fall 2003**

Problem Set 1 Solutions

---

**(E1) (O&W** 1.54)

(a) For the $r = 1$ case, we have:

$$\sum_{n=0}^{n=N-1} 1 = 1 + 1^1 + 1^2 + \cdots + 1^{N-1} = N$$

For the $r \neq 1$ case, by carrying out the long division, we can see that

$$\frac{1}{1-r} = 1 + r + r^1 + r^2 + \cdots + r^{N-1} + \frac{r^N}{1-r} = \sum_{n=0}^{N-1} r^n + \frac{r^N}{1-r}$$

$$\sum_{n=0}^{N-1} r^n = \frac{1-r^N}{1-r}$$

(b) Using the formula we just derived for the $r \neq 1$ case, we have

$$\sum_{n=0}^{\infty} r^n = \lim_{N\to\infty} \sum_{n=0}^{N-1} r^n = \lim_{N\to\infty} \frac{1-r^N}{1-r} = \frac{1}{1-r} - \lim_{N\to\infty} \frac{r^N}{1-r}$$

If $|r| < 1$

$$\lim_{N\to\infty} \frac{r^N}{1-r} = 0$$

So,

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

(c)

$$\begin{aligned}\sum_{n=0}^{\infty} n\alpha^n &= \alpha + 2\alpha^2 + 3\alpha^3 + \cdots \\ &= \alpha\left(1 + 2\alpha + 3\alpha^2 + \cdots\right)\end{aligned}$$

Now we can seperate the contents of the paranthesis on the right-hand-side (RHS) of the equation above as follows:

$$\begin{aligned}\sum_{n=0}^{\infty} n\alpha^n &= \alpha\left(1 + \alpha + \alpha^2 + \cdots + \alpha + 2\alpha^2 + 3\alpha^3 + \cdots\right) \\ &= \alpha\left(1 + \alpha + \alpha^2 + \cdots\right) + \alpha\left(\alpha + 2\alpha^2 + 3\alpha^3 + \cdots\right)\end{aligned}$$

Note that the contents of the second paranthesis on the RHS is the very expression we are trying to evaluate:

$$\begin{aligned}\sum_{n=0}^{\infty} n\alpha^n &= \alpha\left(1 + \alpha + \alpha^2 + \cdots + \sum_{n=0}^{\infty} n\alpha^n\right) \\ (1-\alpha)\sum_{n=0}^{\infty} n\alpha^n &= \alpha\left(1 + \alpha + \alpha^2 + \cdots\right) \\ &= \alpha\sum_{n=0}^{\infty} \alpha^n\end{aligned}$$

Using the result from part (b) for $|\alpha| < 1$,

$$\begin{aligned}(1-\alpha)\sum_{n=0}^{\infty} n\alpha^n &= \frac{\alpha}{1-\alpha} \\ \sum_{n=0}^{\infty} n\alpha^n &= \frac{\alpha}{(1-\alpha)^2}\end{aligned}$$

(d)

$$\begin{aligned}
\sum_{n=k}^{\infty} \alpha^n &= \sum_{n=0}^{\infty} \alpha^n - \sum_{n=0}^{k-1} \alpha^n \\
&= \frac{1}{1-\alpha} - \frac{1-\alpha^k}{1-\alpha} \\
&= \frac{\alpha^k}{1-\alpha}.
\end{aligned}$$

**Problem 1**

(a) For this, we convert the part that is in cartesian form to polar form and proceed from there:

$$\begin{aligned}\angle\left(\sqrt{3}+j\right) &= \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\\ &= \pi/6\\ |\sqrt{3}+j| &= \sqrt{\left(\sqrt{3}\right)^2+1^2}\\ &= 2\\ \sqrt{3}+j &= 2\mathrm{e}^{\pi/6}.\end{aligned}$$

Plugging this into the expression we want to evaluate, we have:

$$\begin{aligned}\left(\sqrt{3}+j\right)^5 \mathrm{e}^{-j\pi/3} &= \left(2\mathrm{e}^{j\pi/6}\right)^5 \mathrm{e}^{-j\pi/3}\\ &= 2^5\mathrm{e}^{j5\pi/6}\cdot\mathrm{e}^{-j\pi/3}\\ &= 32\mathrm{e}^{j\pi/2}\end{aligned}$$

In graphical form, this can be represented as follows:

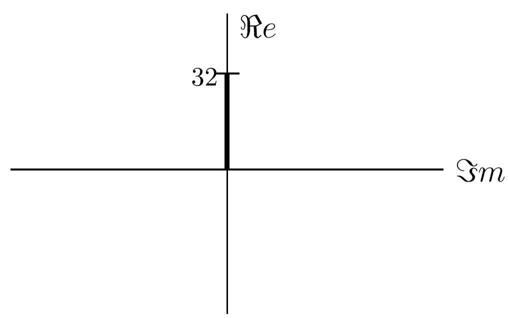



**Figure 1.1a: Magnitude and Phase plot**

**Problem 2**

(a) This problem can be solved in stages. First we flip the signal:

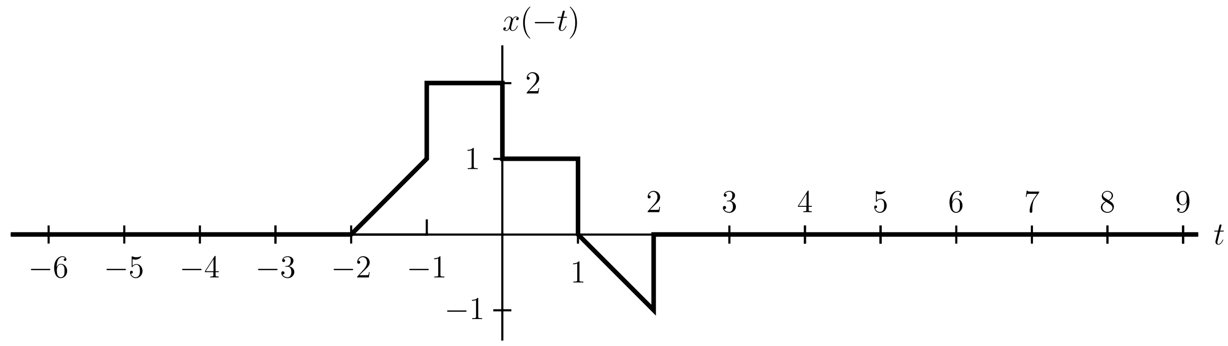



**Figure 2.a.1:** $x(-t)$

Next we scale the time axis by $\frac{1}{3}$:

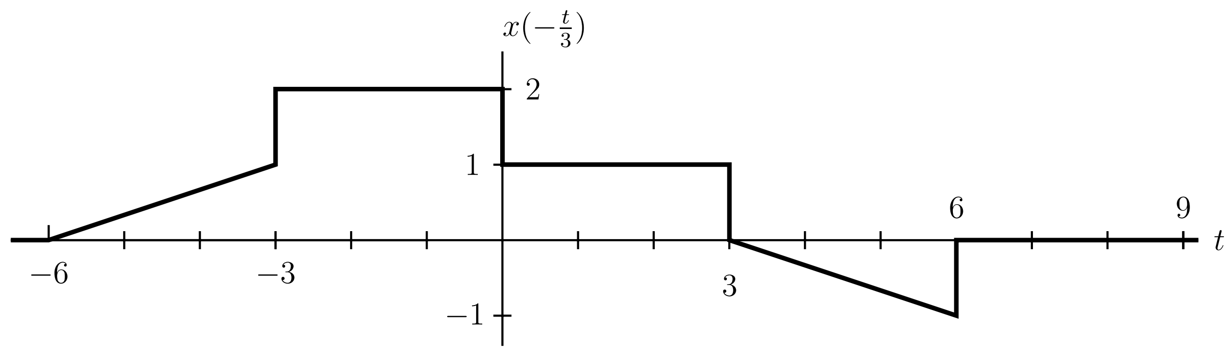



**Figure 2.a.2:** $x(-\frac{t}{3})$

Now we shift by 3 because the time axis has been scaled down by three:

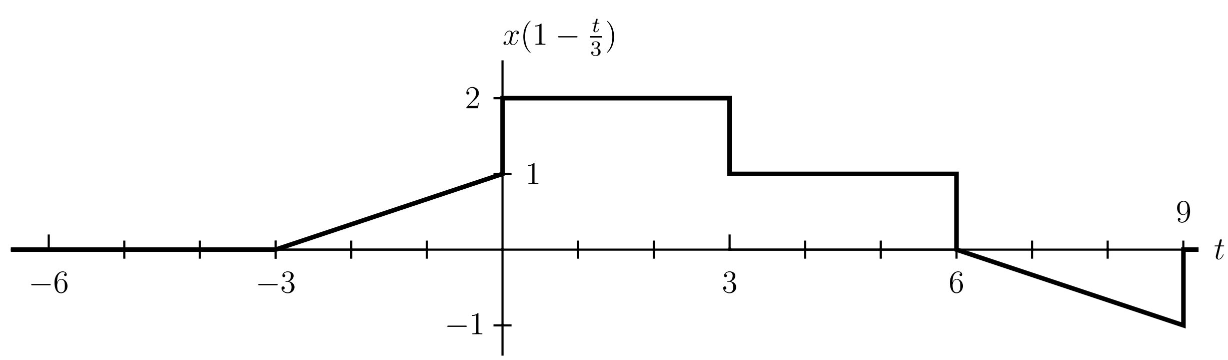



**Figure 2.a.3:** $x(1-\frac{t}{3})$

(b) For this part, we plot the individual signals involved and take the sum of products:

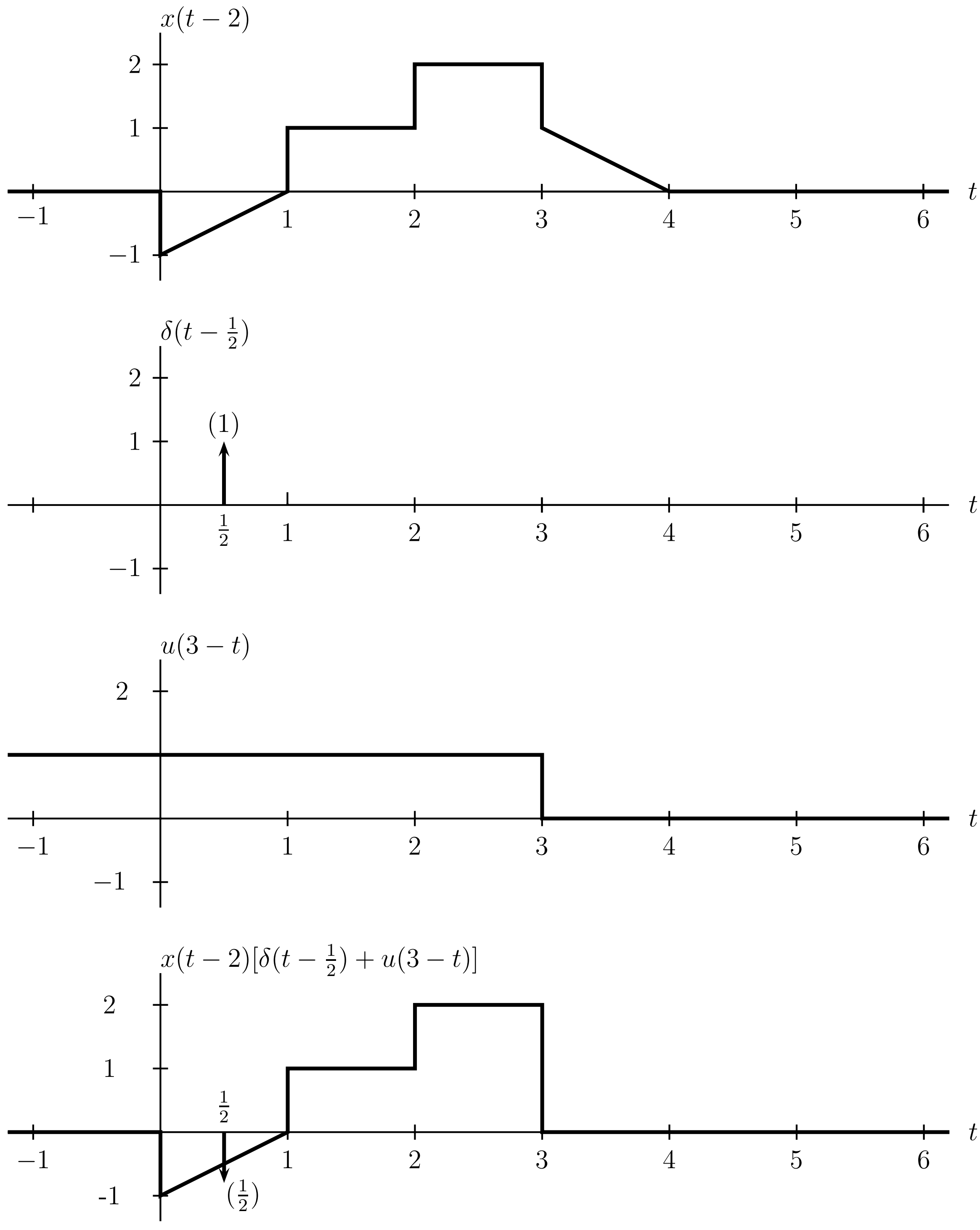
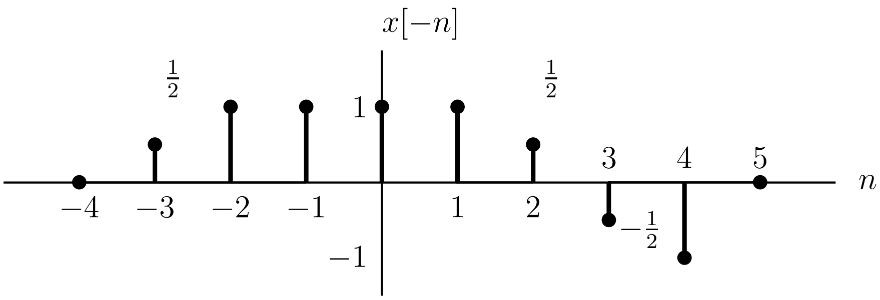
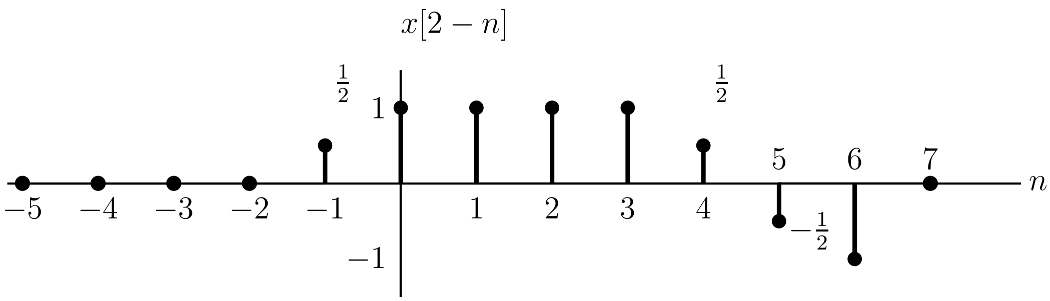



**Figure 2.b:** $x(t-2)[\delta(t-\frac{1}{2})+u(3-t)]$

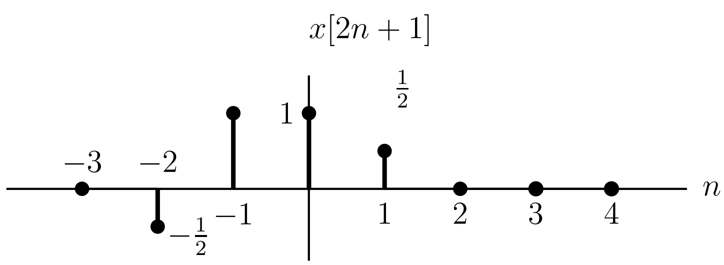
**Problem 3**

(a) This problem can be solved in two stages. First we flip the signal and then we shift by 2:




(b) From the expression, all we have to do is take every odd sample. When you plug in $n = 0, n = 1, n = 2 \cdots$ into the expression $2n + 1$, you end up with the odd samples of the original signal as follows:



**Problem 4**

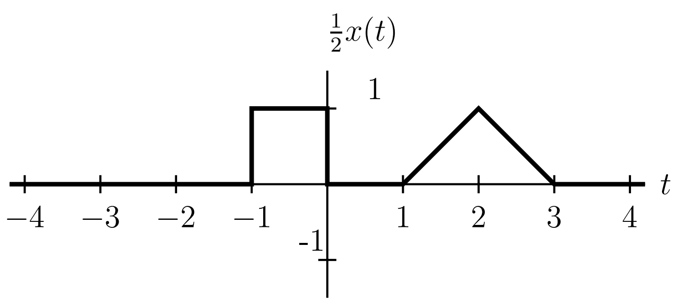
In order to find the even part $x_e(t)$ of a signal $x(t)$, we use the relation:

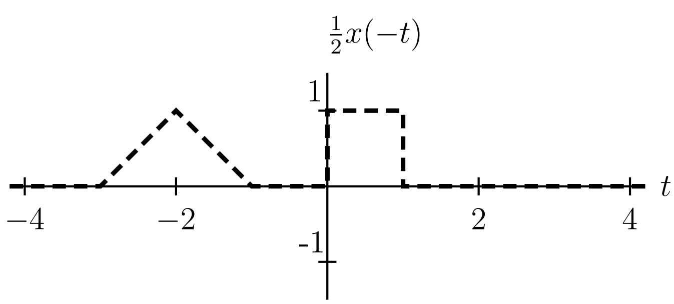
$$x_e(t) = \frac{x(t) + x(-t)}{2}.$$

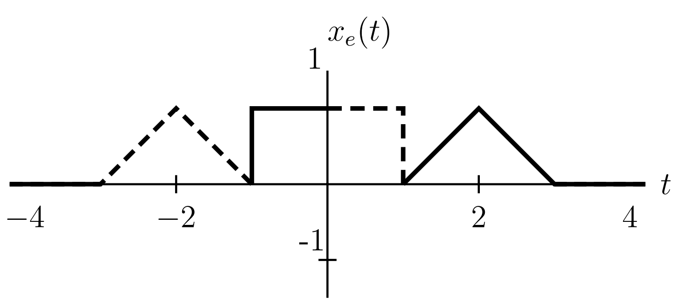
Similarly, for the odd part $x_o(t)$, we use the relation:

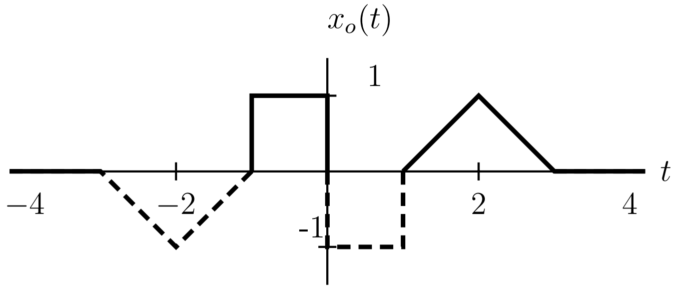
$$x_o(t) = \frac{x(t) - x(-t)}{2}.$$

Using these expressions for the odd and even parts, we end up with the following pictures:






The value of the even part (and the odd part for that matter) at $t = 0$ is ambiguous as it depends on how the plot for $x(t)$ is defined at $t = 0$. The plots in this solution assume that the value of $x(t)$ at $t = 0$ is halfway between 0 and 2, i.e. 1. Using a different definition you may get an even part that is discontinuous at $t = 0$. This is also correct provided it is consistent with your assumption of what the value of $x(t)$ is at the discontinuity. For instance, if you assume that $x(0) = 2$, then the plot of the even part will have a "spike" at $t = 0$ of height 2.

## Problem 5

(a) In order to figure out if a CT signal $x(t)$ is periodic, we need to find a finite, non-zero value of $T$ such that $x(t) = x(t+T)$ for all $t$. The smallest $T$ that satisfies this is the fundamental period.

This function is quite straightforward. We know that the function $\sin(4t-1)$ is periodic with period $\frac{\pi}{2}$. Since the positive and negative cycles of sinusoids have the same shape, the square of this function, i.e. $x(t) = [\sin(4t-1)]^2$ is periodic with fundamental period $\frac{\pi}{4}$.

Also, we can use the relation

$$\sin^2 x = \frac{1}{2} - \frac{1}{2}\sin 2x,$$

which is periodic with period $\frac{\pi}{4}$.

(b) For a DT function $x[n]$, we need to find a finite, non-zero *integer* $N$ such that $x[n] = x[n+N]$ for all $n$. The smallest integer $N$ for which this holds is the fundamental period. If we cannot find such an $N$, then the function is not periodic.

We need

$$\cos\left[4(n+N) + \frac{\pi}{4}\right] \quad = \quad \cos\left[4n + \frac{\pi}{4}\right]$$

For the above to hold, the following has to be true for some integer(s) $k$.

$$\begin{aligned} 4n + 4N + \frac{\pi}{4} &= 4n + \frac{\pi}{4} + 2\pi k \\ N &= \frac{\pi}{2}k \end{aligned}$$

Since $\pi$ is not a rational number, we cannot find an integer N that satisfies this. Thus, the function is not periodic.

(c) We can use the same steps as we did above but we can start with finding the fundamental period of the simpler function $y[n] = \cos\left(\frac{2\pi n}{7}\right)$.

We need the following to hold

$$\cos\left(\frac{2\pi n}{7}\right) = \cos\left(\frac{2\pi(n+N)}{7}\right)$$

So we need the following to hold for at least one integer value of $k$.

$$\begin{aligned} \frac{2\pi n}{7} + 2\pi k &= \frac{2\pi(n+N)}{7} \\ k &= \frac{N}{7} \end{aligned}$$

So, $N = 7, 14, 21, \cdots$ satisfy this.

Now, for $x[n] = (-1)^n \cos\left(\frac{2\pi(n+N)}{7}\right)$ we immediately see that its period has to be an even number because $(-1)^n$ takes on the value of 1 for even $n$ and $-1$ for odd $n$. So, the fundamental period is 14.

**Problem 6**

For this problem, assume that $y_1(t), y_2(t), y_3(t), y_4(t)$ are the outputs of the CT systems when the inputs are $x_1(t), x_2(t), x_3(t), x_4(t)$, respectively. Also, $a$ and $b$ are any (possibly complex) numbers, $t_0$ is any real number and $n_0$ is any integer.

(a) (1) **Memoryless - NO:** Clearly, this is not memoryless because $y(t)$ depends on $x(t+3)$ which is a future value.

(2) **Time-invariant - NO:** Consider the output $y_1(t)$ and a time-shifted version of it $y_1(t+t_0)$ as follows:

$$\begin{aligned} y_1(t) &= x_1(t+3) - x_1(1-t) \\ y_1(t+t_0) &= x_1(t+t_0+3) - x_1(1-t-t_0) \end{aligned}$$

If $x_2(t) = x_1(t+t_0)$ is the input, then the output is given by:

$$\begin{aligned} y_2(t) &= x_2(t+3) - x_2(1-t) \\ &= x_1(t+3+t_0) - x_2(1-t+t_0) \\ &\neq y_1(t+t_0) \end{aligned}$$

Therefore it is not time-invariant

(3) **Linear - YES:**
If $x_3(t) = ax_1(t) + bx_2(t)$. Then,

$$\begin{aligned} y_3(t) &= x_3(t+3) + x_3(1-t) \\ &= ax_1(t+3) + bx_2(t+3) + ax_1(1-t) + bx_2(1-t) \\ &= ax_1(t+3) + ax_1(1-t) + bx_2(t+3) + bx_2(1-t) \\ &= ay_1(t) + by_2(t) \end{aligned}$$

So, this system is linear.

(4) **Causal - NO:** Clearly, this system is not causal because $y(t)$ depends on $x(t+3)$ which is a future value of the input.

(5) **Stable - YES:** Since $y(t)$ is a finite sum of the input $x(t)$ at different time lags, if $x(t)$ is bounded, so is $y(t)$.

(b) (1) **Memoryless - YES:** It is memoryless since $y[n]$ depends only on $x[n]$.

(2) **Time-invariant - NO:**

$$y_1[n] = \begin{cases} (-1)^n x_1[n], & x_1[n] \geq 0 \\ 2x_1[n], & x_1[n] > 0 \end{cases}$$

So, if $x_1[n] = x[n + n_0]$,

$$y_1[n] = \begin{cases} (-1)^n x[n + n_0], & x[n + n_0] \geq 0 \\ 2x[n + n_0], & x[n + n_0] > 0 \end{cases}$$

but,

$$y[n + n_0] = \begin{cases} (-1)^{n+n_0} x[n + n_0], & x[n + n_0] \geq 0 \\ 2x[n + n_0], & x[n + n_0] > 0 \end{cases}$$

So, if $n_0$ is odd, $y[n + n_0] \neq y_1[n]$. Therefore, it is not time-invariant.

(3) **Linear - NO:** Say that $x[0] = 1$, then $y[0] = 1$. Now, if $x_1[0] = -1 \cdot x[0] = -1$, then

$$\begin{aligned} y_1[0] &= -2 \\ &\neq -y[0] \end{aligned}$$

Therefore, it is not linear.

(4) **Causal - YES:** Since the system is memoryless, it is also causal

(5) **Stable - YES** :Any value of $y[n]$ is just a scaled version of the input. So, if $x[n]$ is bounded, so is $y[n]$.

(c) (1) **Memoryless - NO:** It is not memoryless because $y[n]$ depends on the input signal from the time index $n$ to $\infty$.

(2) **Time-Invariant - YES:**
If $x_1[n] = x[n + n_0]$, then

$$\begin{aligned} y_1[n] &= \sum_{k=n}^{\infty} x[k + n_0] \\ &= \sum_{k=n+n_0}^{\infty} x[k] \\ &= y[n + n_0] \end{aligned}$$

Thus, it is time invariant.

(3) **Linear - YES:** Let $x_3[n] = ax_1[n] + bx_2[n]$. So,

$$
\begin{aligned}
y_3[n] &= \sum_{k=n}^{\infty} x_3[n] \\
&= \sum_{k=n}^{\infty} ax_1[n] + \sum_{k=n}^{\infty} bx_2[n] \\
&= ay_1[n] + by_2[n]
\end{aligned}
$$

Thus, the system is linear. Also, since the output is just a sum of the input at different time lags, we can conclude that the system is linear.

(4) **Causal - NO:**
It is not causal because $y[n]$ depends on $x[n], x[n+1] \cdots x[\infty]$.

(5) **Stable - NO:** The output is an infinite sum of the input sequence at time lags of $n \to \infty$. Thus, if the input signal is bounded (e.g. $x[n] = 1$), the output could be unbounded.

**Problem 7**

Since we are dealing with an LTI system, we need to express the input signal $x_2(t)$ in terms of a linear, time-shifted combination of the input whose output is known, i.e. $x_1(t)$ . $x_2(t)$ can be expressed as the sum of $x_a(t) = -2x_1(t-2)$ and $x_b(t) = -x_1(t-1)$ as depicted below:

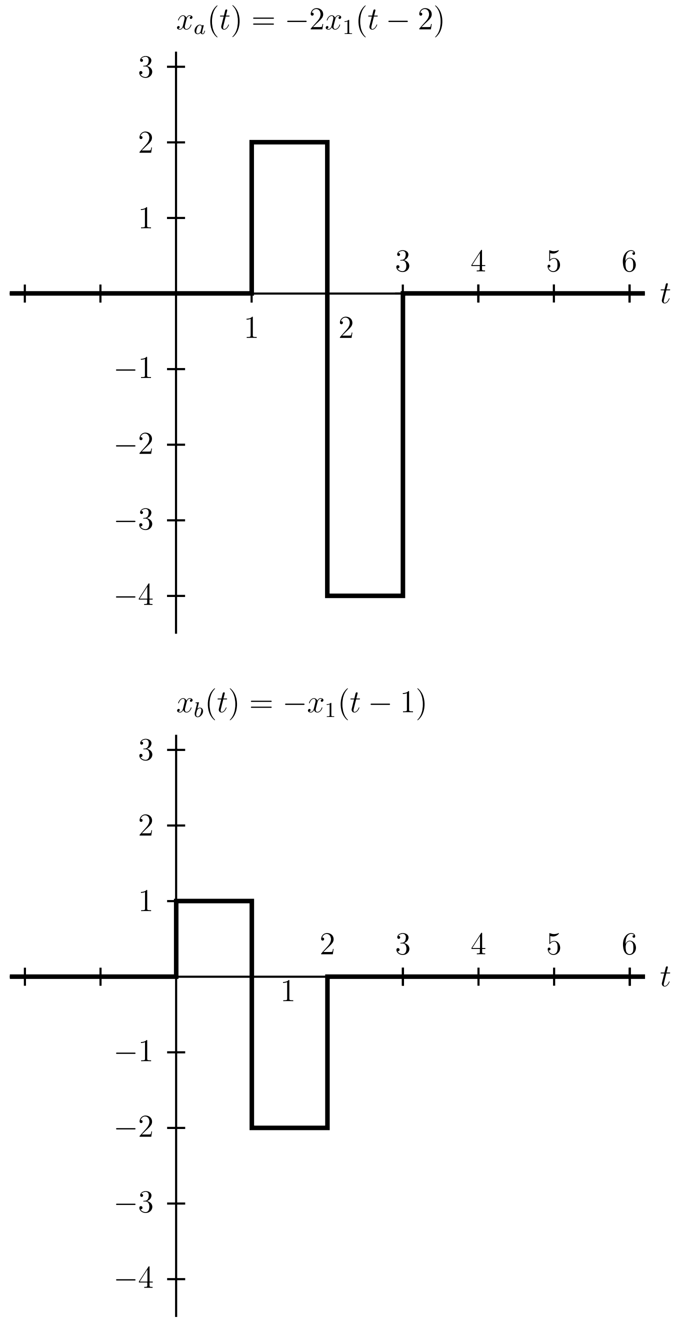



**Figure 7.1:** $x_a(t)$ **and** $x_b(t)$

Let $y_a(t)$ and $y_b(t)$ be the outputs of the system if the inputs are $x_a(t)$ and $x_b(t)$, respectively. From the given input-output pair and using the LTI property, we have the following signals for $y_a(t)$ and $y_b(t)$

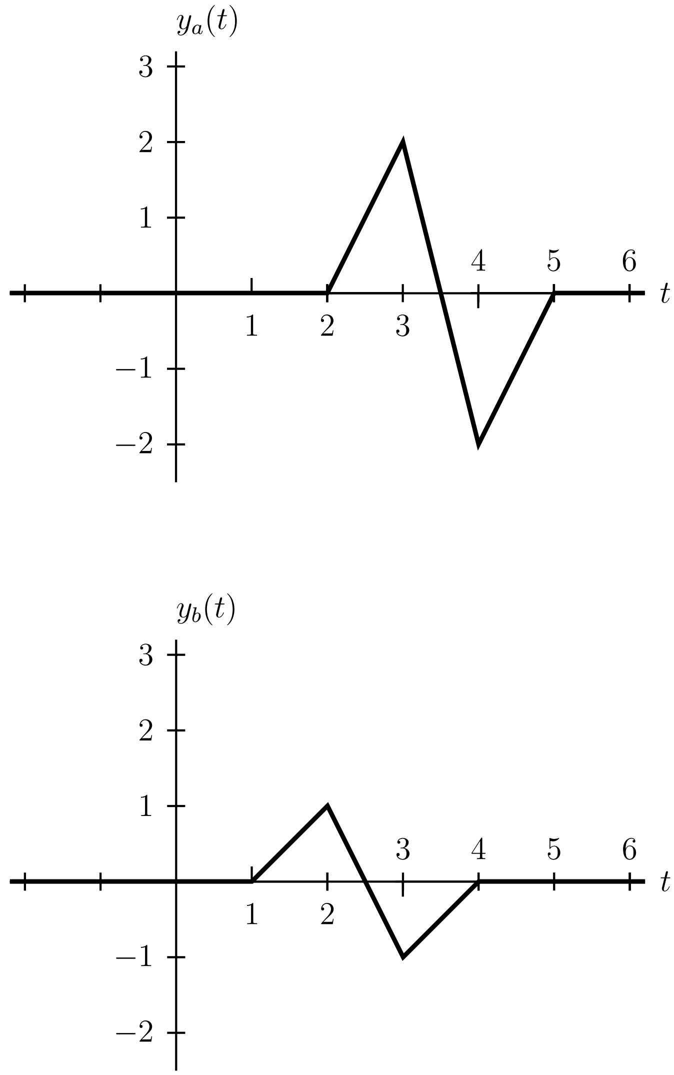



**Figure 7.2:** $y_a(t)$ **and** $y_b(t)$

The sum of these two signals results in the desired output $y_2(t)$:

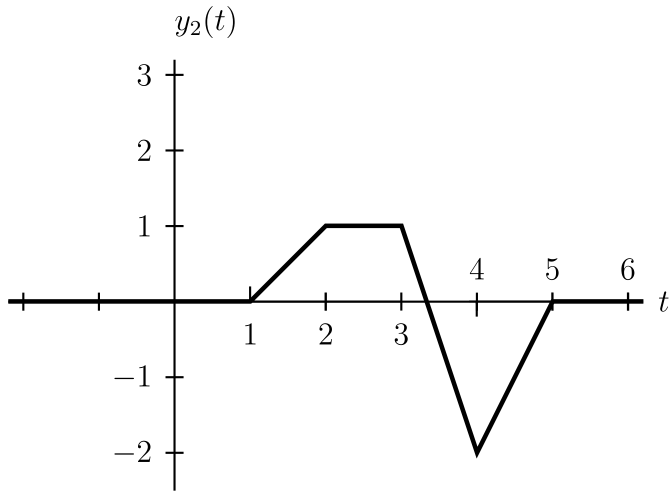



**Figure 7.3:** $y_2(t)$

**Problem 8 (BDS 1.3)** (a) The MATLAB code to complete this exercise is included below.

$$\begin{aligned}
\texttt{nx} &= -3:7; \\
\texttt{x} &= \texttt{zeros}(\texttt{size}(\texttt{nx})); \\
\texttt{x}(4) &= 2; \\
\texttt{x}(6) &= 1; \\
\texttt{x}(7) &= -1; \\
\texttt{x}(8) &= 3;
\end{aligned}$$

Note: index values differ because Matlab does not allow negative index values. Thus, $n = -3$ corresponds to Matlab index 1, so that $\texttt{x}(4) = \texttt{2}$, has the effect of setting $x[0] = 2$ and $\texttt{x}(6) = \texttt{1}$ sets $x[2] = 1$ and so on. (b) Assigning `ny1` through `ny4` can be tricky. The trick is keeping track of which variable is really being plotted. For $y_1[n]$, we want to plot $y_1[\texttt{ny1}]$, but we first need to calculate what `ny1` is, so if you plug `ny1` into $y_1[n]$, we get,

$$y_1[\texttt{ny1}] = x[\texttt{ny1} - 2].$$

Since we know $x[\texttt{nx}]$, we can equate the indices, $\texttt{nx} = \texttt{ny1} - 2$, and solve for `ny1` to get

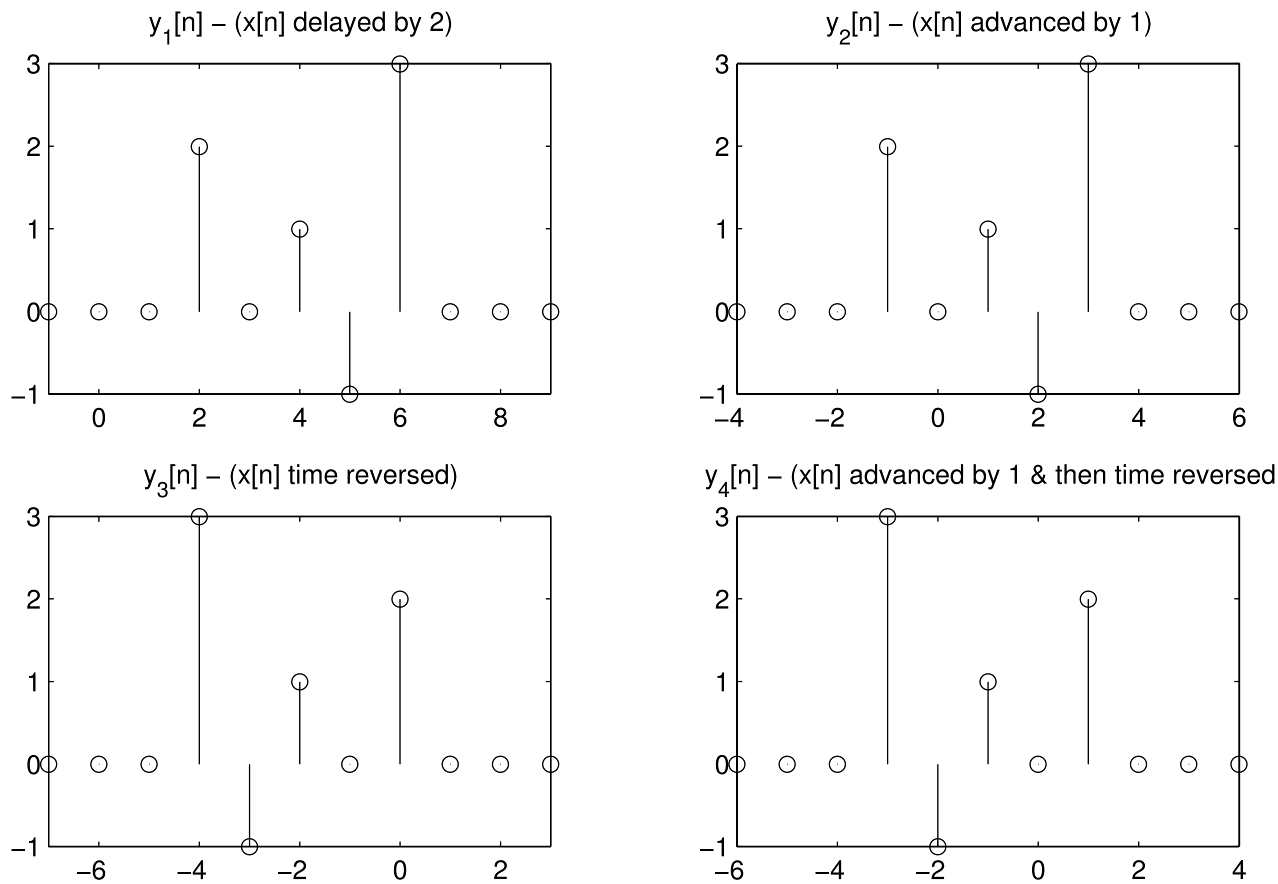
$$\texttt{ny1} = \texttt{nx} + 2.$$

Similarly, one can find that

$$\begin{aligned}
\texttt{ny2} &= \texttt{nx} - 1 \\
\texttt{ny3} &= -\texttt{nx}.
\end{aligned}$$

Finally, `ny4` can be found in a similar manner, but this time we equate $\texttt{nx} = -\texttt{ny4} + 1$. Solving for `ny4` yields

$$\texttt{ny4} = -\texttt{nx} + 1.$$



MATLAB Code

```
% Section 1.3.a

nx = -3:7;
x  = [ 0 0 0 2 0 1 -1 3 0 0 0];

figure(1);
stem(nx, x);


% Section 1.3.b

y1 = x;   ny1 = nx + 2;
y2 = x;   ny2 = nx - 1;
y3 = x;   ny3 = -nx;
y4 = x;   ny4 = -nx + 1;


% Section 1.3.c

figure(2);

subplot(2,2,1);
stem(ny1, y1);
title('y_1[n] - (x[n] delayed by 2)');
axis([ ny1(1) ny1(end) -1 3 ]);

subplot(2,2,2);
stem(ny2, y2);
title('y_2[n] - (x[n] advanced by 1)');
axis([ ny2(1) ny2(end) -1 3 ]);

subplot(2,2,3);
stem(ny3, y3);
title('y_3[n] - (x[n] time reversed)');
axis([ ny3(end) ny3(1) -1 3 ]);

subplot(2,2,4);
stem(ny4, y4);
title('y_4[n] - (x[n] advanced by 1 & then time reversed)');
axis([ny4(end) ny4(1) -1 3]);
```