MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

6.003: Signals and Systems—Fall 2003

Quiz 2
Thursday, November 13, 2003

Directions: The exam consists of 5 problems on pages 2 to 19 and additional work space on pages 20 and 21. Please make sure you have all the pages. Tables of Fourier series properties as well as CT Fourier transform and DT Fourier transform properties and pairs are supplied to you as a separate set of pages. **Enter all your work and your answers directly in the spaces provided on the printed pages of this booklet. Please make sure your name is on all sheets. You may use bluebooks for scratch work, but we will not grade them at all.** All sketches must be adequately labeled. Unless indicated otherwise, **answers must be derived or explained**, not just simply written down. This examination is closed book, but students may use two 8 1/2 × 11 sheets of paper for reference. Calculators may not be used.

NAME: Solutions

Check your section	Section	Time	Rec. Instr.
☐	1	10-11	Prof. Zue
☐	2	11-12	Prof. Zue
☐	3	1- 2	Prof. Gray
☐	4	11-12	Dr. Rohrs
☐	5	12- 1	Prof. Voldman
☐	6	12- 1	Prof. Gray
☐	7	10-11	Dr. Rohrs
☐	8	11-12	Prof. Voldman

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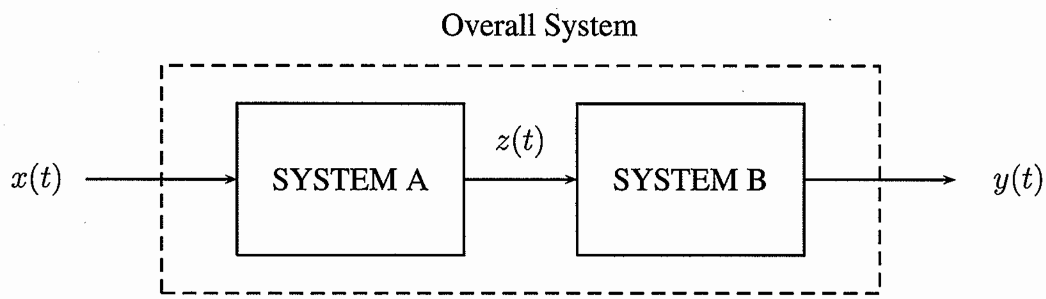
Problem	**No. of points**	**Score**	**Grader**
1	17		
2	21		
3	25		
4	20		
5	17		
Total	100		

PROBLEM 1 (17%)

Consider the following system depicted below:



The input-output relation for SYSTEM A is characterized by the following causal LCCDE:

$$\frac{dz(t)}{dt} + 6z(t) = \frac{dx(t)}{dt} + 5x(t),$$

and the impulse response $h_b(t)$ for SYSTEM B is defined as:

$$h_b(t) = e^{-10t}u(t).$$

Part a. What is the frequency response of the complete system ? That is, given $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$ and $y(t) \overset{\mathcal{F}}{\longleftrightarrow} Y(j\omega)$, determine $H(j\omega) = \dfrac{Y(j\omega)}{X(j\omega)}$.

$$H(j\omega) = \frac{j\omega + 5}{(j\omega + 6)(j\omega + 10)}$$

NAME: Solutions

Work Page for Problem 1

(a)

We can find $H_a(j\omega)$ by inspection, or by using eq. 4.76 (O&W, p. 331):

$$H_a(j\omega) = \frac{j\omega+5}{j\omega+6}$$

We can find $H_b(j\omega)$ using the Basic CTFT Pairs table where:

$$e^{-at}u(t),\ Re\{a\}>0 \longleftrightarrow \frac{1}{a+j\omega}$$

$$h_b(t) = e^{-10t}u(t) \longleftrightarrow H_b(j\omega) = \frac{1}{10+j\omega}$$

$$\Rightarrow H(j\omega) = H_a(j\omega)H_b(j\omega) = \boxed{\frac{j\omega+5}{(j\omega+6)(j\omega+10)} = H(j\omega)}$$

Problem 1 continues on the following page.

Part b. What is the impulse response, $h(t)$ of the complete system ?

$$h(t) = \left(-\frac{1}{4}e^{-6t} + \frac{5}{4}e^{-10t}\right)u(t)$$

Part c. What is the differential equation that relates $x(t)$ and $y(t)$?

$$\frac{d^2y(t)}{dt^2} + 16\frac{dy(t)}{dt} + 60y(t) = \frac{dx(t)}{dt} + 5x(t)$$

Work Page for Problem 1

(b) $H(j\omega) = \dfrac{j\omega+5}{(j\omega+6)(j\omega+10)} = \dfrac{A}{j\omega+6} + \dfrac{B}{j\omega+10}$

$A = H(j\omega)(j\omega+6)\big|_{j\omega=-6} = \dfrac{-6+5}{-6+10} = \boxed{\dfrac{-1}{4} = A}$

$B = H(j\omega)(j\omega+10)\big|_{j\omega=-10} = \dfrac{-10+5}{-10+6} = \dfrac{-5}{-4} = \boxed{\dfrac{5}{4} = B}$

$\rightarrow H(j\omega) = \dfrac{-1/4}{j\omega+6} + \dfrac{5/4}{j\omega+10}$

$\rightarrow \boxed{h(t) = \left(-\frac{1}{4}e^{-6t} + \frac{5}{4}e^{-10t}\right)u(t)}$ (using the pair $e^{-at}u(t),\ Re\{a\}>0 \longleftrightarrow \dfrac{1}{a+j\omega}$)

(c) $H(j\omega) = \dfrac{Y(j\omega)}{X(j\omega)} = \dfrac{j\omega+5}{(j\omega)^2+16j\omega+60}$

by inspection:

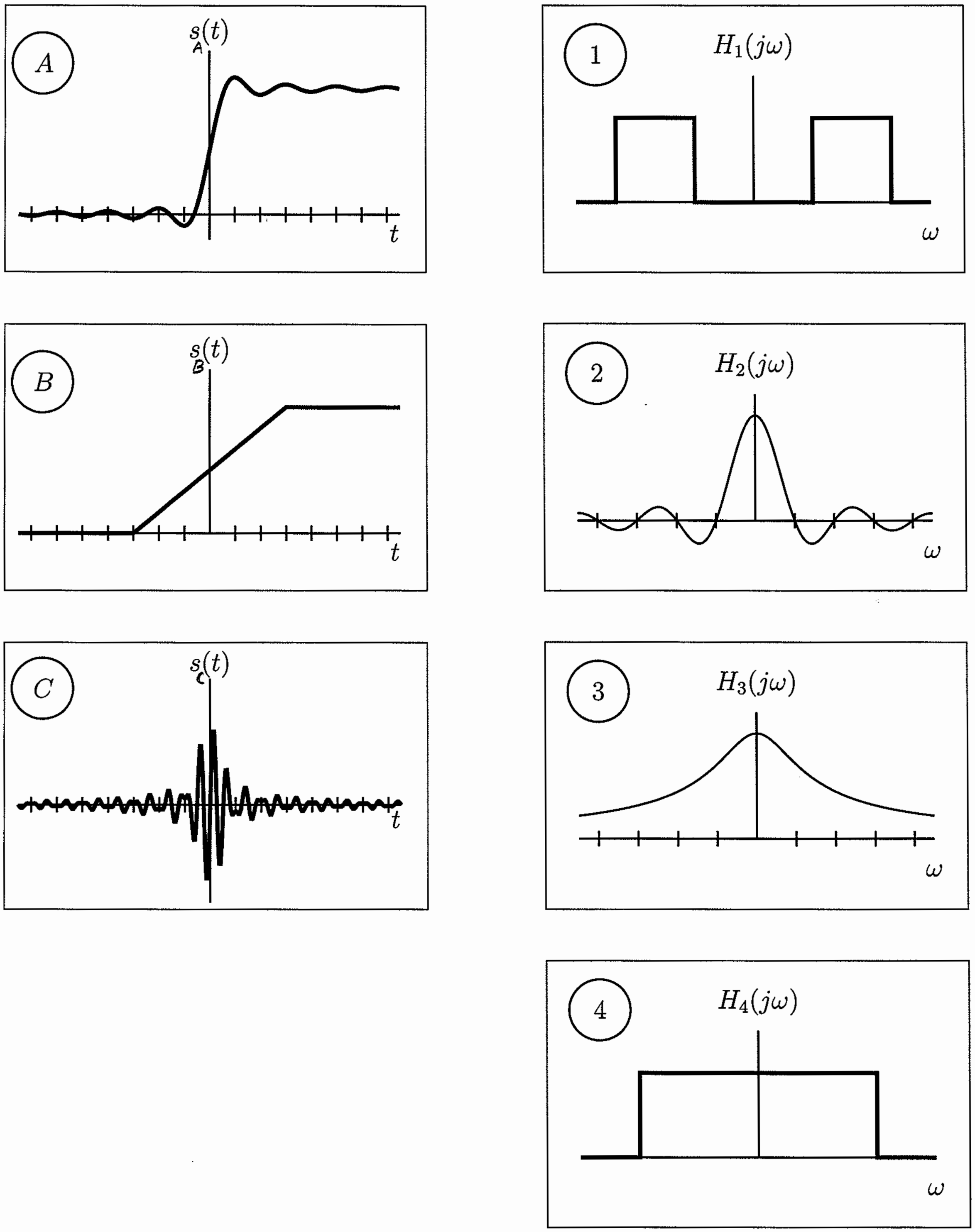
$$\boxed{\frac{d^2y(t)}{dt^2} + 16\frac{dy(t)}{dt} + 60\,y(t) = \frac{dx(t)}{dt} + 5\,x(t)}$$

PROBLEM 2 (21%)

The following images labelled **A, B,** and **C** are step responses corresponding to three of the four frequency responses labelled **1, 2, 3** and **4**. For each of the step responses, please fill in the number of the corresponding frequency response. **For this problem, no explanation or derivation is required.**

Step Response	**Frequency Response**
A	4
B	2
C	1

NAME: Solutions



Work Page for Problem 2

Before matching the step responces with the corresponding frequency responces, let's list few useful expressions:

$$* \; s(t) = \int_{-\infty}^{t} h(\tau)\, d\tau = h(t) * u(t) \longleftrightarrow S(jw) = \frac{H(jw)}{jw} + \pi H(j0)\, \delta(w)$$

$$* \; s(\infty) = \int_{-\infty}^{\infty} h(\tau) d\tau = H(j0)$$

$$* \; x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T \end{cases} \quad (\text{i.e boxcar}) \longleftrightarrow \frac{2 \sin w T_1}{w} \quad (\text{i.e. sinc,})$$

$$* \; \frac{\sin Wt}{\pi t} \quad (\text{i.e. sinc,}) \longleftrightarrow X(jw) = \begin{cases} 1, & |w| < W \\ 0, & |w| > W \end{cases} \quad (\text{i.e boxcar})$$

The reasoning behind each matched pair is the following:

S_B & $H_2(jw)$

$S_B(t)$ looks like a boxcar convolved with $u(t)$, hence the corresponding $h(t)$ is a boxcar. Therefore, the corresponding $H(jw)$ is a sinc function $\Rightarrow H_2(jw)$.

S_C & $H_1(jw)$

$S_C(t)$ seems to approach zero as $t \to \infty$ (i.e $s(\infty) = 0$)

Therefore, the corresponding $H(jw)$ has a zero DC gain (i.e. $H(j0) = 0$)

$\Rightarrow H_1(jw)$

Work Page for Problem 2

There is also another way to see that $S_c(t)$ & $H_1(j\omega)$ are a pair:

$S_c(t)$ is a real an odd function $\longrightarrow$ $S_c(j\omega)$ is purely imaginary and odd.

$$\because S_c(j\omega) = \frac{H(j\omega)}{j\omega} + \pi H(j0)\delta(t)$$

$\therefore$ $H(j\omega)$ must be real and even & $H(j0) = 0$

$\Rightarrow$ $H_1(j\omega)$ is the corresponding frequency responce.

$S_A(t)$ & $H_4(j\omega)$

$S_A(t)$ has two properties that will enable us to find the corresponding $H(j\omega)$:

* $S_A(\infty) \neq 0 \Rightarrow H(j0) \neq 0$
* $S_A(t)$ is oscillating $\Rightarrow h(t)$ must be oscillating too.

Possible causes for oscillation in $h(t)$ are:

* Peaking at a non-zero frequency (for example a BPF).
* Discontinuity in any derivatives of $H(j\omega)$ (think about the Gibbs Phenomena, or about an impulse in the frequency domain corresponding to a complex exponential in the time domain)

$H_3(j\omega)$ has a non-zero DC-gain, but is a smooth function and doesn't have any non-DC peaking $\Rightarrow$ it can't be the corresponding $H(j\omega)$.

$H_4(j\omega)$ is a boxcar which has a non-zero DC-gain and its corresponding impulse responce is a sinc (i.e. has oscillations)

$\Rightarrow H_4(j\omega)$ is the corresponding frequency responce.

PROBLEM 3 (25%)

Part a. Determine the Fourier transform $R(e^{j\omega})$ of the following sequence:

$$r[n] = \begin{cases} 1, & 0 \le n \le M,\ M \text{ is an even integer greater than 1} \\ 0, & \text{otherwise.} \end{cases}$$

$$R(e^{j\omega}) = \frac{e^{-j\omega\frac{M}{2}} \sin\left[\omega\left(\frac{M+1}{2}\right)\right]}{\sin(\omega/2)}$$

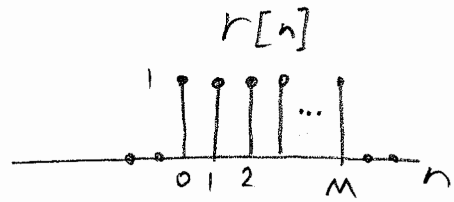
Part b. Consider the sequence

$$w[n] = \begin{cases} \frac{1}{2}\left(1 - \cos\left(\frac{2\pi n}{M}\right)\right), & 0 \le n \le M \\ 0, & \text{otherwise,} \end{cases}$$

where M is as defined in **Part a.** Express $W(e^{j\omega})$, the Fourier transform of $w[n]$ in terms of $R(e^{j\omega})$, the Fourier transform of $r[n]$ above.

$$W(e^{j\omega}) = \frac{1}{2}R(e^{j\omega}) - \frac{1}{4}R\left(e^{j(\omega - \frac{2\pi}{M})}\right) - \frac{1}{4}R\left(e^{j(\omega + \frac{2\pi}{M})}\right)$$

Work Page for Problem 3



(a) From the figure, we can see that $r[n]$ is an even boxcar shifted by $\frac{M}{2}$ to the right (notice that $\frac{M}{2}$ is integer because M is even).

Using the time shifting property and the following DTFT Pair: $x[n] = \begin{cases} 1, & |n| \leq N_1 \\ 0, & |n| > N_1 \end{cases} \longleftrightarrow \frac{\sin[\omega(N_1+\frac{1}{2})]}{\sin(\omega/2)}$

we find:

$$\boxed{R(j\omega) = e^{-j\omega\frac{M}{2}} \frac{\sin[\omega(\frac{M+1}{2})]}{\sin(\omega/2)}}$$

(b) $w[n]$ can be expressed as:

$$w[n] = r[n] \cdot \left[\frac{1}{2} - \frac{1}{2}\cos\left(\frac{2\pi n}{M}\right)\right]$$
$$= r[n] \cdot \left[\frac{1}{2} - \frac{1}{4}e^{\frac{j2\pi n}{M}} - \frac{1}{4}e^{-j\frac{2\pi n}{M}}\right]$$
$$= \frac{1}{2}r[n] - \frac{1}{4}r[n]e^{\frac{j2\pi n}{M}} - \frac{1}{4}r[n]e^{-j\frac{2\pi n}{M}}$$

using the frequency shifting property:

$$\rightarrow \boxed{W(e^{j\omega}) = \frac{1}{2}R(e^{j\omega}) - \frac{1}{4}R\left(e^{j(\omega-\frac{2\pi}{M})}\right) - \frac{1}{4}R\left(e^{j(\omega+\frac{2\pi}{M})}\right)}$$

 Problem 3 continues on the following page.

Part c. Is there a positive even integer M that will make $W(e^{j\omega})$ real ? If so, find the values of M that satisfy this constraint. If not, explain why.

YES

Values of M ____________________

$\boxed{\textbf{NO}}$

Explanation:

For $W(e^{jw})$ to be real, $w[n]$ needs to be even.

The value of M only determines the width of $w[n]$ which has the value of zero for $n<0$.

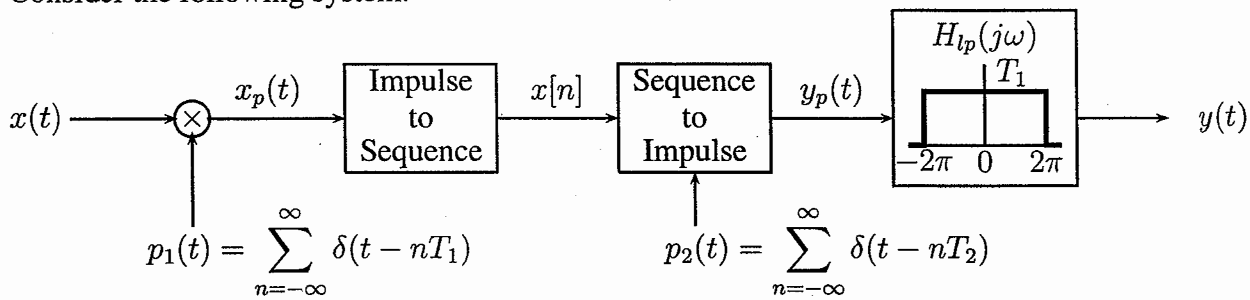
$\rightarrow$ $w[n]$ can't be made even by changing M

$\rightarrow$ there is no value of M that would make $W(e^{jw})$ real.

Work Page for Problem 3

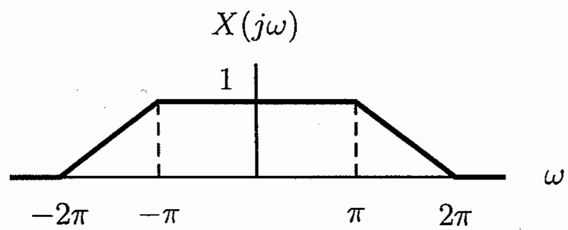
PROBLEM 4 (20%)

Consider the following system:



$p_1(t)$ is an impulse train whose fundamental period is T_1 and $p_2(t)$ is another impulse train whose fundamental period is T_2. $H_{lp}(j\omega)$ is a lowpass filter whose gain is T_1 and cutoff frequency is at ω_c. Note that $x[n] = x(nT_1)$ and $y_p(t) = \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT_2)$.

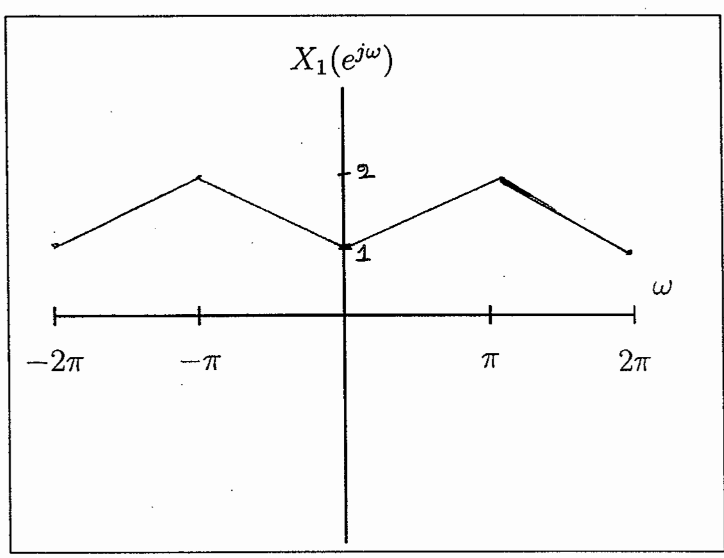
The input $x(t)$ is a band limited real signal whose Fourier transform is shown below:

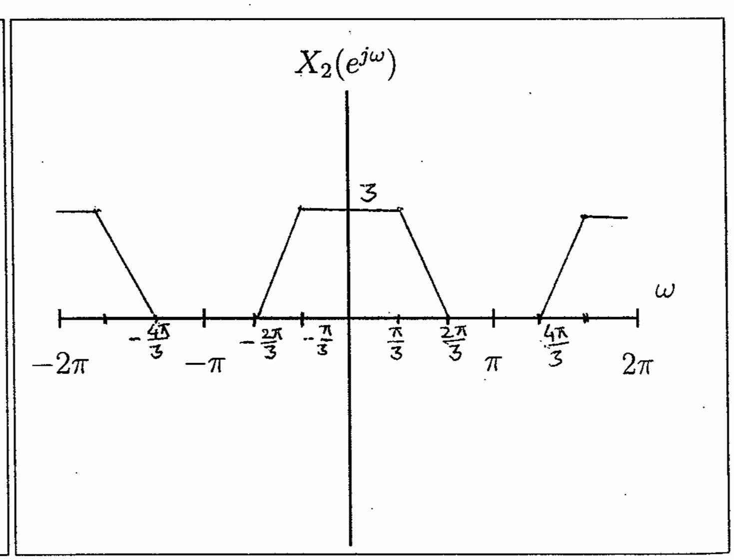


Part a. Let's define

$$
\begin{aligned}
x_1[n] &= x(nT_1), \quad \text{where } T_1 = 1, \\
x_2[n] &= x(nT_1), \quad \text{where } T_1 = \frac{1}{3}.
\end{aligned}
$$

In the given axes below provide the labeled sketches of $X_1(e^{j\omega})$ and $X_2(e^{j\omega})$, Fourier transforms of $x_1[n]$ and $x_2[n]$ respectively.

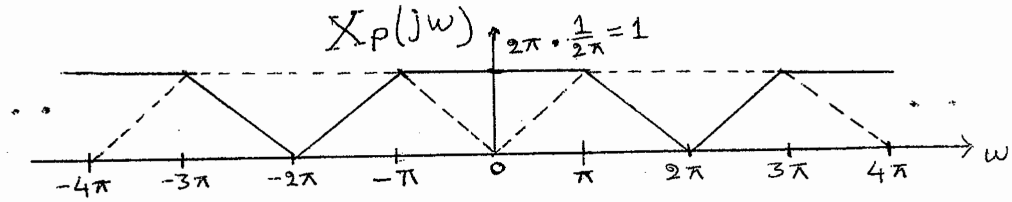




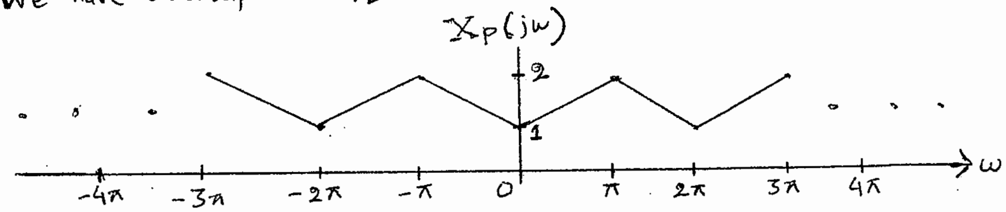
Work Page for Problem 4

$p_1(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_1)$ and $T_1 = 1$. $P_1(j\omega)$ is an impulse train with fundamental period $\frac{2\pi}{T_1} = 2\pi$ and area under each impulse is 2π.

$x_p(t) = x(t) \cdot p_1(t)$ Using the multiplication property and taking Fourier transfor

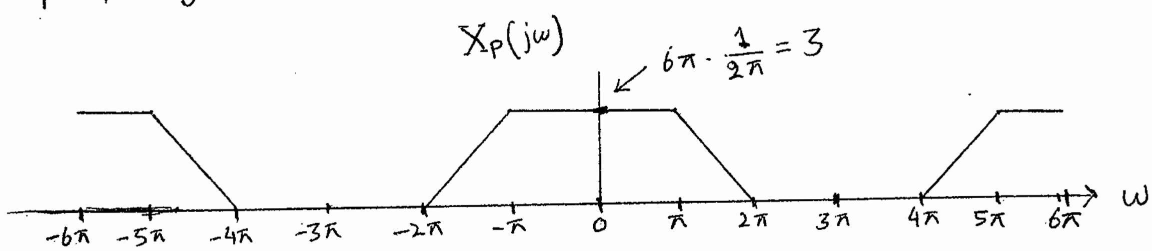


We have overlap as Nyquist rate for sampling is not satisfied. Adding the overlap regions



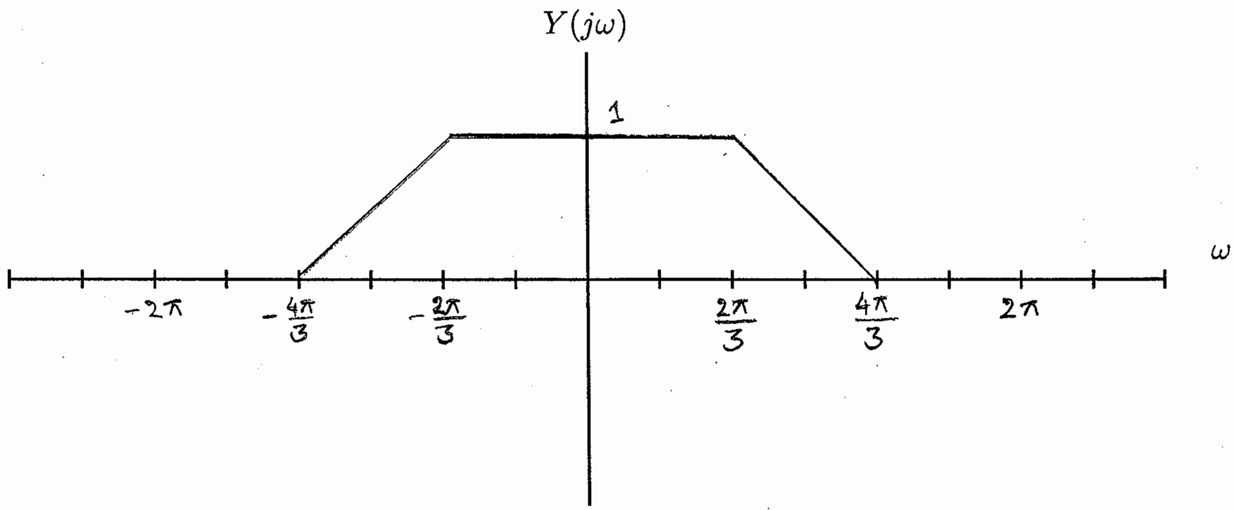
To find $X_1(e^{j\Omega})$, we use $\Omega = \omega T_1$ and scale the frequency axis of $X_p(j\omega)$ by $T_1 = 1$. Resulting $X_1(e^{j\Omega})$ is shown on page 14.

Next, $T_1 = \frac{1}{3}$. Therefore, $P_1(j\omega)$ is periodic with fundamental period $\frac{2\pi}{T_1} = 6\pi$ and each impulse has an area of 6π. Using the multiplication property again,



Multiplying the frequency axis of $X_p(j\omega)$ by $T_1 = \frac{1}{3}$, we get $X_2(e^{j\Omega})$ as shown on page 14.

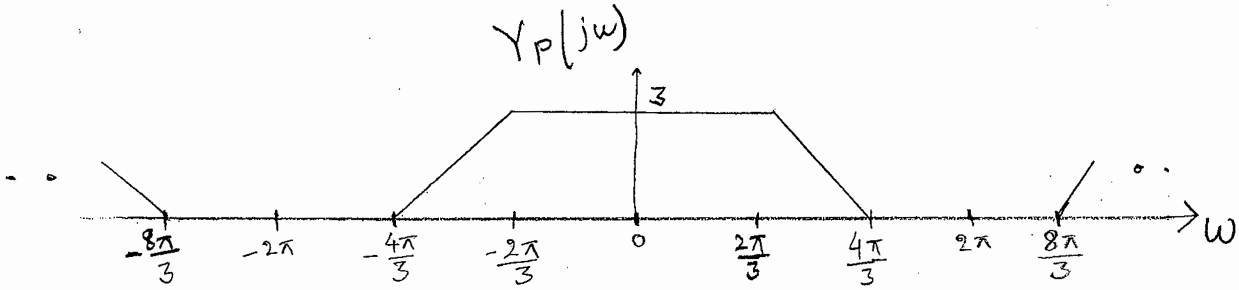
Part b. Suppose $T_1 = \frac{1}{3}$ and $T_2 = \frac{1}{2}$. Provide a labeled sketch of $Y(j\omega)$, Fourier transform of the overall output $y(t)$.



Work Page for Problem 4

Here $T_1 = \frac{1}{3}$. Therefore, $X(e^{j\Omega}) = X_2(e^{j\Omega})$ as found in part (a).

To find $Y_p(jw)$, we scale the frequency axis of $X(e^{j\Omega})$ by $\frac{1}{T_2} = 2$.

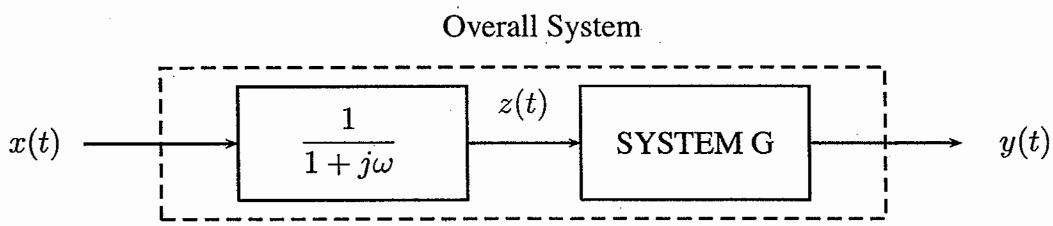


$Y_p(jw)$ goes through a low pass filter with cutoff frequency $w_c = 2\pi$ and gain $T_1 = \frac{1}{3}$

Therefore, resulting $Y(jw)$ is as shown on page 16.

PROBLEM 5 (17%)

We have a cascade of two stable CT LTI systems as shown below:

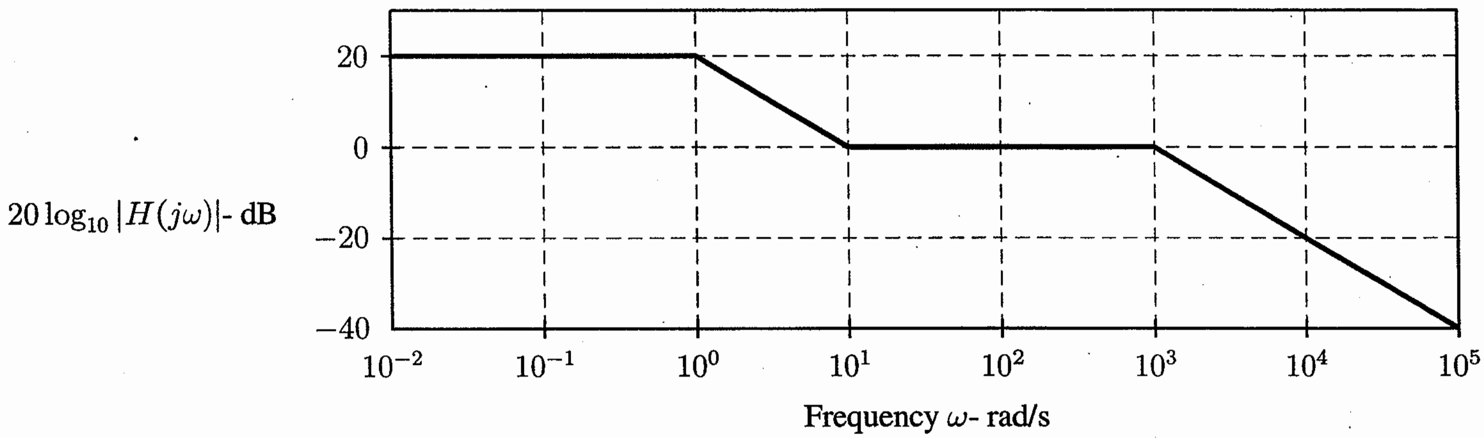


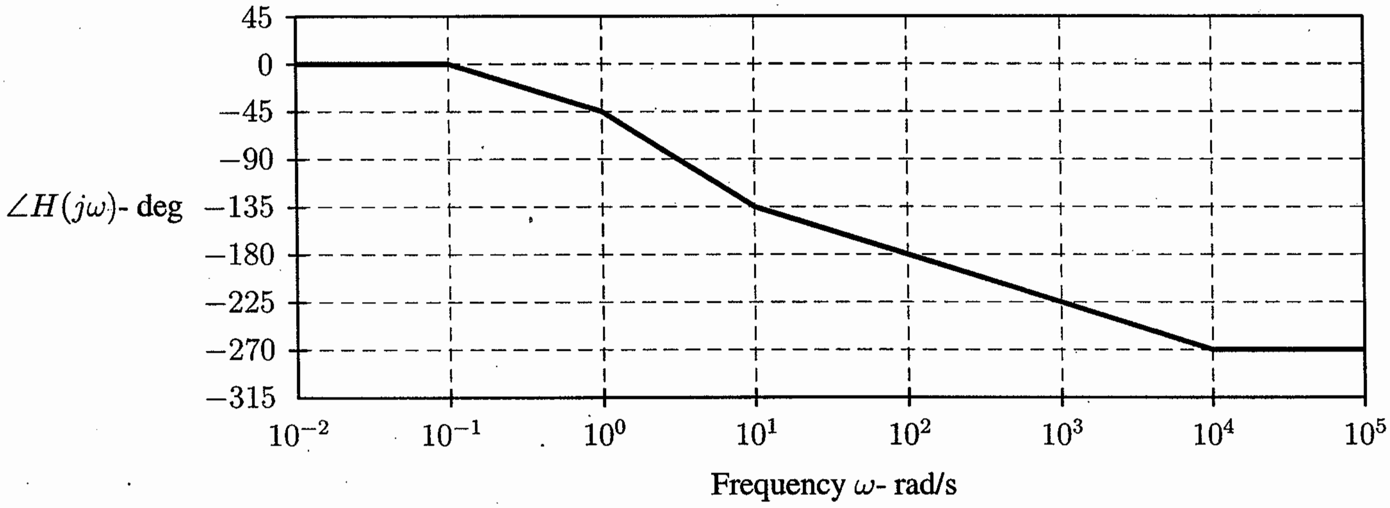
The straight line approximation of Bodé plots of the overall system, $H(j\omega)$ is shown in the next page.

Find the frequency response, $G(j\omega)$, of SYSTEM G.

$$G(j\omega) = \frac{10\,(1 - j\omega\, 0.1)}{1 + j\omega\, 0.001}$$

NAME:________________________________


20
0
−20
−40
$20\log_{10}|H(j\omega)|$- dB
10^{-2} 10^{-1} 10^{0} 10^{1} 10^{2} 10^{3} 10^{4} 10^{5}
Frequency ω- rad/s


45
0
−45
−90
−135
−180
−225
−270
−315
$\angle H(j\omega)$- deg
10^{-2} 10^{-1} 10^{0} 10^{1} 10^{2} 10^{3} 10^{4} 10^{5}
Frequency ω- rad/s

Work Page

From the magnitude plot, we see $H(j\omega)$ has 3 breakpoints.

$$H(j\omega) = \frac{1}{1+j\omega} \cdot G(j\omega)$$

Therefore, $G(j\omega)$ needs 2 breakpoints.

let. $G(j\omega) = C \cdot G_1(j\omega) \cdot G_2(j\omega)$

$G_1(j\omega)$ has a breakpoint at $\omega=10$ and magnitude plot of $G_1(j\omega)$ needs to go up starting at $\omega=10$ with 20 dB/dec slope. From the phase plot of $H(j\omega)$, we see the phase goes down with slope $\frac{\pi}{2}$/dec starting at $\omega=\frac{10}{10}=1$

Therefore, $G_1(j\omega) = 1 - j\omega \cdot \frac{1}{10} = 1 - j\omega\, 0.1$

$G_2(j\omega)$ has a breakpoint at $\omega=10^3$ and magnitude plot of $G_2(j\omega)$ needs to go down starting at $\omega=10^3$ with slope 20 dB/dec. From the phase plot of $H(j\omega)$ we see the phase goes down with slope $\frac{\pi}{4}$/dec starting at $\omega=\frac{10^3}{10}=10^2$.

Therefore, $G_2(j\omega) = \dfrac{1}{1+j\omega\frac{1}{10^3}} = \dfrac{1}{1+j\omega\, 0.001}$

Work Page

Finally, $C = 10$ makes the magnitude plot of $H(j\omega)$ to start at 20 dB.

Therefore,

$$G(j\omega) = 10 \cdot (1 - j\omega\, 0.1) \cdot \frac{1}{1 + j\omega\, 0.001}$$